Spin Foams: Foundations

自旋泡沫: 基础

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Spin foams provide a framework for the dynamics of loop quantum gravity that is manifestly covariant under the full four-dimensional diffeomorphism symmetry group of general relativity. In this way, they complete the ideal of three-dimensional diffeomorphism covariance that consistently motivates loop quantum gravity at every step. Specifically, spin foam models aim to provide a projector onto, and a physical inner product on, the simultaneous kernel of all of the constraints of loop quantum gravity by means of a discretization of the gravitational path integral. In the limit of small Planck constant, they are closely related to the path integral for Regge calculus while at the same time retaining all of the tools of a canonical quantum theory of gravity. They may also be understood as generalizations of well-understood state sum models for topological quantum field theories. In this chapter, we review all of these aspects of spin foams, as well as review in detail the derivation of the currently most used spin foam model, the EPRL model, calculational tools for it, and the various extensions of it in the literature. We additionally summarize some of the successes and open problems in the field.

自旋泡沫为圈量子引力提供了一个动力学框架，该框架明显满足广义相对论完整四维微分同胚对称群的协变性。以此方式，它实现了三维微分同胚协变的理想，而这一理想始终是圈量子引力每一步发展的一致动机。具体而言，自旋泡沫模型旨在通过引力路径积分离散化，为圈量子引力所有约束的共同零空间提供投影算子与物理内积。在普朗克常数很小的极限下，它与里奇微积分的路径积分密切相关，同时保留了正则量子引力理论的全部工具。它也可以被理解为拓扑量子场论中已知状态和模型的推广。本章我们综述自旋泡沫的所有上述方面，同时详细推导目前应用最广的自旋泡沫模型——EPRL 模型，介绍该模型的计算工具，以及文献中对该模型的各类拓展。我们还额外总结了该领域的部分成果与待解决问题。

Keywords

关键词

Spin foam formalism - Loop quantum gravity - Spin networks - Quanta of space - Covariant quantum gravity

自旋泡沫形式论 - 圈量子引力 - 自旋网络 - 空间量子 - 协变量子引力

Introduction

引言

Loop quantum gravity gives a compelling description of what quantum space may look like: a collection of atoms of space, whose geometry shows all the characteristics of a quantum system, such as discrete spectra, incompatibility of classical observables, superpositions, and entanglement. The crucial question for the theory is how to describe the dynamics of such quanta, in a way that is both ultraviolet complete and compatible with general relativity in the appropriate limit. The spin foam formalism is an approach to the dynamics based on constructing a well-defined version of the gravitational path integral with boundary states that belong to the Hilbert space of loop quantum gravity. Excellent reviews and introductions to this formalism exist in the literature; see [1-5]. Our choice for this chapter was to present something complementary and up-to-date with respect to them. In this chapter, we go over the historical path that motivated the tools used in the formalism, starting from the idea that one should define transition amplitudes for spin networks that implement all quantum constraints in a covariant way. We then focus on a specific model, the Lorentzian EPRL model, which is both the currently most studied model and also rich enough to allow us to introduce all types of techniques and ideas that are also used in other models. We conclude with an overview on the current open questions and research directions.

圈量子引力对量子空间的可能形态给出了令人信服的描述: 量子空间是由空间原子构成的集合，其几何展现出量子系统的全部特征, 例如离散谱、经典可观测量的不相容性、叠加与纠缠。该理论的关键问题是，如何以一种紫外完备且在合适极限下与广义相对论相容的方式描述这些量子的动力学。自旋泡沫形式是处理动力学的一种方案，它基于构造一个定义良好的引力路径积分版本，其边界态属于圈量子引力的希尔伯特空间。现有文献中已经有关于该形式的优秀综述与导论；参见 [1-5]。本章我们选择呈现相对于这些内容具有补充性且最新的内容。在本章中，我们梳理推动该形式所用工具发展的历史路径，出发点是如下理念: 我们应当为协变实现全部量子约束的自旋网络定义跃迁振幅。随后我们聚焦于一个具体模型——洛伦兹 EPRL 模型，它既是目前研究最多的模型，也足够丰富，能够让我们介绍同样适用于其他模型的各类技术与理念。最后我们概述当前的开放问题与研究方向。

The Spin Foam Gist

自旋泡沫要义

The Path Integral as a Means to Implement the Constraints of Gravity

作为实现引力约束方法的路径积分

Loop quantum gravity is a canonical approach to quantum gravity, with a Hilbert space of states and operators corresponding to the basic variables of the theory. As a canonical approach, from the very start, it treats space and time very differently, a fact very much at odds with the spirit of both special and general relativity, in which space and time are part of a single unified continuum. Feynman's path integral approach opened the door to a manifestly spacetime covariant formulation of quantum mechanics. It is therefore natural to ask: Is there a path integral version of loop quantum gravity, and if so, what does it look like? The quest to answer this question leads to the spin foam formalism.

圈量子引力是量子引力的正则方法，拥有对应理论基本变量的态希尔伯特空间与算符。作为一种正则方法，它从一开始就对空间和时间作出完全不同的处理，这一点与狭义和广义相对论的核心精神相悖——狭义与广义相对论中，空间和时间是统一连续统的组成部分。费曼路径积分方法为量子力学提供了显式时空协变表述。因此人们很自然会问: 圈量子引力是否存在路径积分版本？如果存在，它是什么样的？回答这个问题的研究衍生出了自旋泡沫形式。

General relativity is a theory with constraints: in addition to equations of motions which determine time evolution, there are also equations of motion that are independent of time, constraining the data at any fixed time. Such equations of motions are called constraints; when cast in the form of a set of phase space functions C_i set to zero, the phase space functions C_i are likewise referred to as constraints. Familiar examples of constrained systems include Maxwell theory, general relativity, and Yang-Mills theory. The theory of the quantization of constrained systems was formalized by Dirac [6]. Each constraint C_i whose Poisson brackets with other constraints are again constraints is called first class and leads to a quantum equation $\widehat{C}_i\psi = 0$ which supplement Schrödinger's equation. Solutions to the quantum constraints are called physical states, and the space of all such states, equipped with an appropriate inner product, is called the physical Hilbert space.

广义相对论是一个带约束的理论: 除了决定时间演化的运动方程外，还存在不依赖时间的运动方程，会对任意固定时刻的初始数据施加约束。这类运动方程被称为约束；当约束被表示为一组置零的相空间函数 C_i 时，这些相空间函数 C_i 也同样被称为约束。人们熟悉的约束系统例子包括麦克斯韦理论、广义相对论和杨-米尔斯理论。约束系统的量子化理论由狄拉克 [6] 形式化。任意与其他约束的泊松括号仍为约束的约束 C_i 被称为第一类约束，它会导出补充薛定谔方程的量子方程 $\widehat{C}_i\psi = 0$。量子约束的解被称为物理态，所有这类态构成的空间配备合适内积后，被称为物理希尔伯特空间。

The first-class constraints of a theory are in one-to-one correspondence with its gauge symmetries: In order for initial data to uniquely determine a full solution to the equations of motion, it is necessary that any two phase space points related, by the flow generated by a first-class constraint, be considered the same physical state. That is, the flow generated by each first-class constraint must be considered a family of gauge transformations. In quantum theory, the flow generated by each first-class constraint operator $\widehat{C}_i$ is the family of transformations $\exp\left(i\lambda\widehat{C}_i\right)$. When this family acts on a physical state, $\widehat{C}_i\psi = 0$, its action is trivial, $\exp\left(i\lambda\widehat{C}_i\right)\psi = \psi$, consistent with its interpretation as a family of gauge transformations also at the quantum level. In fact, at the quantum level, the condition that ψ satisfy the quantum constraint equations is equivalent to requiring that it be gauge invariant. One can distinguish between gauge transformations which preserve the polarization of the quantization from those that don't. In the first case, the transformation has a well-defined action on the argument of the wave function ψ and acts on ψ via pull-back, while in the second case, the action on ψ is more complicated.

一个理论的第一类约束与其规范对称性一一对应: 要让初始数据唯一确定运动方程的完整解，任何由第一类约束生成流联系起来的两个相空间点，都必须被视为同一个物理态。也就是说，每个第一类约束生成的流都应当被看作一族规范变换。在量子理论中，每个第一类约束算符 $\widehat{C}_i$ 生成的流就是变换族 $\exp\left(i\lambda\widehat{C}_i\right)$。当该变换族作用在物理态 $\widehat{C}_i\psi = 0$ 上时，其作用是平凡的 $\exp\left(i\lambda\widehat{C}_i\right)\psi = \psi$，这与它在量子层面作为规范变换族的诠释一致。实际上，在量子层面，ψ 满足量子约束方程的条件等价于要求它是规范不变的。我们可以区分保持量子化极化的规范变换和不保持极化的规范变换。在前一种情况中，变换对波函数 ψ 的宗量有良定义作用，并且通过拉回作用在 ψ 上；而在后一种情况中，对 ψ 的作用要更复杂。

The basic principle giving rise to general relativity is the principle of general covariance, which is equivalent to covariance under the group of four-dimensional diffeomorphisms, the basic symmetry group of the theory. These both are also equivalent to Einstein's deeper principle of background independence that there be no background spacetime structure in physical law, but only dynamical spacetime structures. This basic symmetry group is furthermore a gauge symmetry: Its action on initial data is exactly what is generated by the main constraints of the theory, the Hamiltonian and diffeomorphism constraints, which, respectively, generate time evolution with respect to an arbitrary time coordinate and spatial diffeomorphisms. For spatially compact regions, the Hamiltonian of the theory is furthermore a linear combination of the constraints, so that, in quantum theory, physical states do not evolve in time - the famous so-called problem of time, directly related to the fact that the choice of time coordinate in general relativity is a pure gauge choice.

产生广义相对论的基本原理是广义协变原理，它等价于四维微分同胚群下的协变性，这是该理论的基本对称群。二者还等价于爱因斯坦更深层次的背景独立性原理: 物理规律中不存在背景时空结构，只有动力学时空结构。这个基本对称群还是规范对称: 它对初始数据的作用正是由该理论的主约束——哈密顿约束和微分同胚约束生成，它们分别生成对应任意时间坐标的时间演化，以及空间微分同胚。对于空间紧致区域，该理论的哈密顿量本身就是约束的线性组合，因此在量子理论中，物理态不随时间演化——这就是著名的“时间问题”，它直接源于广义相对论中时间坐标的选择是纯粹的规范选择这一事实。

Just as the framework of canonical quantization needed to be generalized by Dirac, Bergmann, and others to accommodate the conceptual novelties of general relativity, so too, in order to apply Feynman's derivation of the path integral formalism to quantum gravity, it must be generalized, specifically in three different ways:

正因为正则量子化框架需要狄拉克、伯格曼等人推广才能容纳广义相对论的新概念，要将费曼的路径积分形式推导应用到量子引力，也必须对其做推广，具体来说要从三个不同方向推广:

1. The path integral for an unconstrained system has the interpretation as the probability amplitude for a given initial quantum state to evolve into a final quantum state after a specified amount of time. It thereby encodes the quantum time evolution of the system - the Schrödinger equation. For constrained systems, there are quantum equations of motion besides Schrödinger's equation, and the path integral in this case encodes them as well: The path integral between a given initial and final state provides the probability amplitude for the projection of the initial state onto the physical Hilbert space to evolve into the final state after a given amount of time. This is true for any constrained system. For constraints generating non-compact orbits, physical states are non-normalizable - that is, distributional - and the projector onto physical states is then defined in a generalized sense. For general relativity, because additionally quantum time evolution is trivial, no time evolution is involved in the interpretation of the path integral, and it formally corresponds to the probability amplitude for the projection of the initial state into the physical Hilbert space to be found in the final state. That is, it provides the physical inner product between the projections of the initial and final states into the physical Hilbert space and in this way encodes the dynamics of the theory. The new role of the gravitational path integral as projector on to physical states was noted by Hartle and Hawking [7] and later in more detail by Halliwell and Hartle [8].

1. 无约束系统的路径积分可诠释为: 给定初始量子态经过指定时间演化到末量子态的概率幅。因此它编码了系统的量子时间演化——即薛定谔方程。对于约束系统，除薛定谔方程外还存在其他量子运动方程，而路径积分在这种情况下同样会对这些方程进行编码: 给定初态和末态之间的路径积分，给出初始态投影到物理希尔伯特空间后经过指定时间演化到末态的概率幅。这一点对任意约束系统都成立。对于生成非紧轨道的约束，物理态不可归一化——即属于分布类，因此物理态投影算符需要在推广的意义下定义。对于广义相对论，此外量子时间演化本身是平凡的，因此路径积分的诠释不涉及时间演化，它形式上对应初始态投影到物理希尔伯特空间后出现在末态的概率幅。也就是说，它给出了初态和末态各自投影到物理希尔伯特空间后的物理内积，以此方式编码了理论的动力学。引力路径积分作为物理态投影算符的这一新作用由 Hartle 和 Hawking 最先指出 [7]，之后 Halliwell 和 Hartle 进行了更详细的研究 [8]。

2. If one considers Feynman's derivation more carefully, one sees that, instead of integrating over classical histories, one must integrate over histories of eigenvalues of a complete set of operators, obtained from inserting resolutions of the identity at infinitesimal time intervals. In the case in which the spectra of the op-

erators are continuous, this reduces to the more familiar case of an integral over classical histories. But in the case of loop quantum gravity, this is not the case, and so one sums over a certain class of discrete spacetime geometries, called spin foams, which motivates the name for the formalism.

2. 如果更仔细地考察费曼的推导，就会发现，我们并不是对经典历史积分，而是需要对一组完备算子的本征值构成的历史积分，这组本征值来自无穷小时间间隔处插入的单位分解。当算子的谱是连续谱时，这就退化为我们更熟悉的对经典历史积分的情况。但在圈量子引力中，情况并非如此，因此我们对一类特定的离散时空几何求和，这类几何被称为自旋泡沫，这也是该形式体系名称的由来。

3. In background-dependent field theories, path integrals provide transition amplitudes between initial and final states. This is achieved by integrating over histories in a region bounded by two Cauchy slices and assuming suitable falloff conditions at spatial infinity. The initial and final states then belong to the Hilbert spaces associated with the initial and final slices, and if first-class constraints are present, the path integral automatically projects on the physical Hilbert space; see, e.g., [8,9]. In a background-independent theory, the split into Cauchy slices has no preferred physical meaning. What one can do is to avoid it altogether and instead apply the formalism to a general bounded region. The result gives a quantum amplitude for the 3-geometry on this compact boundary which is formally diffeomorphism invariant and background independent. This formulation of quantum field theory is called the general boundary formulation. It was present in the early ideas of the Hartle-Hawking proposal and was formally developed by Oeckl [10] and applied to the LQG context starting with [2,11]; see also [12] for an early review. It is central to all concrete work in spin foams that has been done since. The resulting quantum amplitude has a ”frozen time” status; hence, some interpretive work is needed to use it to describe physical processes. A useful procedure is to use partial observables, namely, splitting the system to introduce a physical clock and re-express the amplitude as a function of the clock reading [2].

3. 在依赖背景的场论中，路径积分给出初态和末态之间的跃迁振幅。这通过对介于两个柯西面之间的区域中的历史积分，并假设空间无穷远满足合适的衰减条件来实现。初态和末态分属初始柯西面和末柯西面对应的希尔伯特空间，如果存在第一类约束，路径积分会自动投影到物理希尔伯特空间；参见例如 [8,9]。在不依赖背景的理论中，分解为柯西面不具有优先的物理意义。我们可以完全规避这种分解，转而将该形式体系应用于一般有界区域。最终得到紧致边界上 3-几何的量子振幅，它形式上是微分同胚不变且背景独立的。这种量子场论表述被称为一般边界表述，它早已出现在 Hartle-Hawking 方案的早期思想中，后由 Oeckl 进行了形式化发展 [10]，从文献 [2,11] 开始被应用到圈量子引力框架中；早期综述参见 [12]。它对于此后所有自旋泡沫方向的具体研究都至关重要。得到的量子振幅具有“冻时间”属性，因此需要做一些诠释工作才能用它描述物理过程。一个实用方法是使用部分可观测量，即将系统拆分来引入物理时钟，再将振幅表示为时钟读数的函数 [2]。

With these generalizations, application to loop quantum gravity of Feynman’s path integral leads to the spin foam formalism. Spin foams are not the only existing formalism for a gravitational path integral - there are a number of precursors, notably the Euclidean path integral formalism of Gibbons and Hawking [13], quantum Regge calculus [14], and causal dynamical triangulations [15-17]. What makes the spin foam formalism unique is that it provides transition amplitudes (or, equivalently, a projector onto physical states with physical inner product) for a canonical quantum theory of gravity with mathematically well-defined inner product and operators. In this way, the spin foam formalism is able to overcome corresponding limitations of other approaches. At the same time, the spin foam formalism strives to respect the basic principles of general relativity: diffeomorphism covariance or background independence and that gravity is geometry.

经过这些推广后，将费曼路径积分应用到圈量子引力就得到了自旋泡沫形式体系。自旋泡沫并不是引力路径积分唯一现存的形式体系——它有许多先驱，最著名的是 Gibbons 和 Hawking 的欧几里得路径积分形式体系 [13]、量子 Regge 微积分 [14] 以及因果动力学三角剖分 [15-17]。自旋泡沫形式体系的独特之处在于，它为正则量子引力提供了跃迁振幅 (或者等价地说，提供了带有物理内积的物理态投影算符)，且其内积和算子在数学上都是良定义的。通过这种方式，自旋泡沫形式体系能够克服其他方法的相应局限。同时，自旋泡沫形式体系始终遵循广义相对论的基本原理: 微分同胚协变性即背景独立性，以及引力就是几何这一核心思想。

In addition, the spin foam formalism turns out to include topological quantum field theories - that is, quantum field theories with no local physical degrees of freedom that are therefore simpler and well understood - that pre-date the spin foam formalism. These early topological spin foams are important as toy models for both understanding the four-dimensional gravitational case and providing building blocks for the construction of the latter, as we will see.

此外，人们发现自旋泡沫形式体系包含了拓扑量子场论——即不存在局域物理自由度的量子场论，这类理论更简单也更容易理解，且出现得比自旋泡沫形式体系更早。这些早期的拓扑自旋泡沫作为玩具模型十分重要，既可以帮助我们理解四维引力的情况，也可以作为构造模块来构建四维引力的自旋泡沫模型，我们在后文中会看到这一点。

Transition Amplitudes for Spin Network States

自旋网络态的跃迁振幅

Let us elaborate on the points raised above in mathematical terms. Consider first perturbative quantum gravity at the linear order, in which everything is under control and one can see explicitly that the path integral projects on physical states. Using the flat background as reference, one can compute the path integral between an initial and a final configuration $h_{1,2}$ separated by a time interval T , given by

让我们从数学层面详细阐述上述观点。首先考虑线性阶的微扰量子引力，该框架下一切都可控，我们可以清晰地看到路径积分向物理态投影。以平直背景为参考，我们可以计算间隔为 T 的初末构型 $h_{1,2}$ 之间的路径积分，其表达式为

$$K\left[h_1, h_2, T\right] := \int_{h(0,\mathbf{x})=h_1}^{h(T,\mathbf{x})=h_2} \mathcal{D}h e^{iS(h)}. \tag{1}$$

The integration implements the constraints and gives a sum over energy eigenstates of the free theory [9],

该积分实现了约束，并给出了自由理论能量本征态的求和 [9],

$$K\left[h_1, h_2, T\right] = \sum_n e^{-iE_n T} \overline{\Psi}_n \left[h_1^{\mathrm{TT}}\right] \Psi_n \left[h_2^{\mathrm{TT}}\right], \tag{2}$$

where TT denotes the transverse-traceless parts of the spatial metric. At the linear level, all spacetime diffeomorphisms preserve the polarization, so that the meaning of gauge invariance of the wave function is manifest from its argument, as discussed earlier.

其中 TT 表示空间度规的横向无迹部分。在线性层面，所有时空微分同胚都保持极化，因此正如前文讨论，波函数规范不变性的含义可由其宗量直接体现。

Moving on to a background-independent approach, both the flat metric on the Cauchy slices and the time separation T on the time-like boundary are now part of the dynamical boundary data. Let us denote by q the whole classical information on the boundary. The path integral (1) should be replaced by a formal expression like

转向背景独立方法后，柯西面上的平直度规和类时边界上的时间间隔 T 现在都成为动力学边界数据的一部分。我们用 q 表示边界上的全部经典信息，原路径积分 (1) 应替换为如下形式表达式:

$$K[q] := \int_{g|_{\partial M}=q} \mathcal{D}g e^{iS(g)}. \tag{3}$$

This is the essence of the general boundary formalism: It associates a quantum amplitude to q as a whole. Only after a physical clock is introduced, and an associated split of the boundary, can this be meaningfully interpreted as a transition amplitude between initial and final physical states. To turn this formal expression into a well-defined mathematical formula, we seek to make $K[q]$ precise such that it provides a quantum amplitude for the states of loop quantum gravity.

这就是一般边界形式的核心: 它将量子振幅与整个 q 关联起来。只有引入物理时钟并对边界做相应拆分后，我们才能将其有意义地解释为初末物理态之间的跃迁振幅。为了将这个形式表达式转化为定义良好的数学公式，我们需要明确 $K[q]$，使其能够为圈量子引力的态提供量子振幅。

Loop quantum gravity, being based on a connection formulation of general relativity, has three sets of first-class constraints: the Gauss constraint, generating local gauge rotations of the SU(2) principal fiber bundle, the diffeomorphism constraint, and the Hamiltonian constraint. Because the gauge symmetries generated by the first two of these have a well-defined action on the connection - the argument of the kinematical wave function - their implementation on the kinematical Hilbert space is unambiguous and straightforward. Recall that each normalizable state of the kinematical Hilbert space is labeled by a graph and this corresponds to a truncation of the number of degrees of freedom. This truncation corresponds to a distributional description of the gravitational field, but can also be interpreted in terms of discrete, piece-wise flat geometries; see, e.g., [18] and Sect. 4 of this Handbook's on - Chap. 83, "Emergence of Riemannian Quantum Geometry." Due to the truncated nature of the normalizable states, the gauge group generated by the quantum Gauss constraint is in a precise sense compact, so that its solutions are a subspace of the kinematical Hilbert space, with orthonormal basis given by the spin network states $|S\rangle$. Each of these states is labeled by an embedded graph, an assignment of an SU(2) irreducible representation to each link, and to each node an intertwiner among the representations labeling the adjacent links.

圈量子引力建立在广义相对论的联络表述基础上，包含三类第一类约束: 生成 SU(2) 主纤维丛局域规范转动的高斯约束、微分同胚约束和哈密顿约束。由于前两类生成的规范对称性对运动学波函数的宗量——联络——有定义清晰的作用，它们在运动学希尔伯特空间上的实现是明确且直接的。我们知道，运动学希尔伯特空间的每个可归一化态都由一个图标记，这对应对自由度数量的截断。该截断对应引力场的分布描述，但也可以解释为离散分片平坦几何；参见例如文献 [18] 和本手册第 83 章“黎曼量子几何的涌现”第 4 节。由于可归一化态的截断性质，量子高斯约束生成的规范群在严格意义上是紧致的，因此其解构成运动学希尔伯特空间的一个子空间，其正交归一基由自旋网络态 $|S\rangle$ 给出。每个这类态都由一个嵌入图标记: 给每条边分配一个 SU(2) 不可约表示，给每个节点分配相邻边表示之间的缠结算。

The gauge group generated by the diffeomorphism constraint, by contrast, is non-compact, and so the solutions are distributional. A projector from spin network states to these distributional solutions, and inner product on the resulting image, is defined by averaging over the action $U(\varphi)$ of diffeomorphisms [19]:

与之相对，微分同胚约束生成的规范群是非紧致的，因此其解是分布性的。从自旋网络态到这些分布解的投影，以及投影结果上的内积，由对微分同胚作用 $U(\varphi)$ 做平均定义 [19]:

$$\Big|s(S)\rangle := \int \mathcal{D}\varphi \Big|\varphi \cdot S\rangle, \tag{4}$$

$$\begin{aligned}\langle s(S'), s(S)\rangle &:= \int \mathcal{D}\varphi\mathcal{D}\varphi' \langle \varphi' \cdot S' \mid \varphi \cdot S\rangle = \int \mathcal{D}\varphi\mathcal{D}\varphi' \left\langle S' \left| U\left(\varphi'^{-1} \circ \varphi\right)\right| S\right\rangle \\ &= \int \mathcal{D}\varphi \langle S' | U(\varphi) | S\rangle = \langle S' \mid s(S)\rangle = \langle s(S') \mid S\rangle \end{aligned} \tag{5}$$

where in going from the first to the second line of (5), $\mathcal{D}\varphi$ has been assumed to be left-invariant and have unit volume. The averaged spin network $s(S)$ is labeled by a spin network ”up to diffeomorphism.” Although information about knotting of the graph survives averaging over diffeomorphisms, all known observables, as well as the dynamics considered in this chapter, are insensitive to this information, and so one can, up to some subtleties [19-21], think of $s(S)$ as a spin network on an abstract graph. Note the final two expressions in (5) are well-defined functions of $s(S)$ and $s(S')$ and, with appropriate assumptions on the measure $\mathcal{D}\varphi$ [19], are finite. Thus, though the projection in (4) is not normalizable in the kinematical inner product, it is normalizable in the inner product defined in (5). The resulting Hilbert space is denoted $\mathcal{H}_{\text{Diff}}$.

在 (5) 从第一行到第二行的推导中，我们假设 $\mathcal{D}\varphi$ 是左不变的，且体积为单位 1。平均后的自旋网络 $s(S)$ 由“差一个微分同胚”的自旋网络标记。尽管图的纽结信息在微分同胚平均后保留下来，但所有已知可观测量以及本章讨论的动力学都不依赖这一信息，因此忽略一些细节 [19-21] 后，我们可以将 $s(S)$ 视为抽象图上的自旋网络。注意 (5) 中最后两个表达式是定义良好的 $s(S)$ 和 $s(S')$ 的函数，且在对测度 $\mathcal{D}\varphi$ 做适当假设下 [19] 是有限的。因此，尽管 (4) 中的投影在运动学内积下不可归一化，它在 (5) 定义的内积下是可归一化的，得到的希尔伯特空间记为 $\mathcal{H}_{\text{Diff}}$ 。

One can then proceed to construct a projector from $\mathcal{H}_{\text{Diff}}$ to solutions of the Hamiltonian constraint again by averaging over the corresponding gauge, similar to (4) [22, 23]. The situation is however more complicated. We sketch here the procedure and its shortcomings, because it serves both as a useful review and as a motivation for spin foams. If $\widehat{H}(x)$ denotes the Hamiltonian constraint operator, and $\widehat{H}[N]$ its smearing by an arbitrary lapse $N(x)$, we have

随后，我们可以如 (4) 那样 [22, 23]，再次通过对相应规范求平均，构造从 $\mathcal{H}_{\text{Diff}}$ 到哈密顿约束解的投影算子。但情况要更复杂。我们在此概述该方法及其不足，因为它既是有用的回顾，也是自旋泡沫的研究动机。若 $\widehat{H}(x)$ 为哈密顿约束算符，$\widehat{H}[N]$ 是它对任意移距 $N(x)$ 的抹除形式，我们有

$$P\Big|s\rangle := \prod_x P_x\Big|s\rangle = \prod_x \int_{-\infty}^{\infty} dN(x)\, e^{-iN(x)\widehat{H}(x)}\Big|s\rangle = \int \mathcal{D}N e^{-i\widehat{H}[N]}\Big|s\rangle$$
$$= \int \mathcal{D}\varphi \int \mathcal{D}N e^{-i\widehat{H}[N]}\Big|\varphi\cdot S\rangle = \int \mathcal{D}\varphi U(\varphi) \int \mathcal{D}N e^{-i\widehat{H}[\varphi^* N]}\Big|S\rangle$$
$$= \int \mathcal{D}\varphi U(\varphi) \int \mathcal{D}N e^{-i\widehat{H}[N]}\Big|S\rangle = \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int \mathcal{D}\varphi U(\varphi)\int \mathcal{D}N\widehat{H}[N]^m\Big|S\rangle$$

(6)where P_x denotes the projector onto the kernel of $\widehat{H}(x)$ and S is any kinematical spin network that averages to s . In the last step, the diffeomorphism invariance of the measure $\mathcal{D}N$ was used. Note the last three expressions above are manifestly diffeomorphism invariant, in addition to formally satisfying the Hamiltonian constraint. One thus expects P to project from $\mathcal{H}_{\text{Diff}}$ into solutions of all the constraints.

其中 P_x 表示投影到 $\widehat{H}(x)$ 核的投影算子，S 是任意运动学自旋网络，平均后得到 s 。最后一步用到了测度 $\mathcal{D}N$ 的微分同胚不变性。注意上述最后三个表达式除了形式上满足哈密顿约束外，还明显具有微分同胚不变性。因此我们预期 P 会将 $\mathcal{H}_{\text{Diff}}$ 投影到所有约束的解空间中。

To continue the derivation, at this point, one must choose a specific proposal for the Hamiltonian constraint operator. More than one such proposal exists. In the end, as we wish to simply to motivate a general ansatz, this choice does not matter too much, and so we consider the so-called ”Euclidean” term in the first one, that introduced by Thiemann in [24]. Thiemann's smeared Hamiltonian constraint operator $\widehat{H}[N]$ is defined so that it acts only at nodes of the spin network and includes an holonomy operator that introduces a sum over spins. The action of the Euclidean part takes the expression

为了继续推导，此时必须选择一个具体的哈密顿约束算符方案。这类方案不止一种。归根结底，由于我们只是想推导出一个一般假设，这个选择无关紧要，因此我们讨论第一个方案中所谓的“欧几里得”项，即 Thiemann 在文献 [24] 中引入的方案。Thiemann 的抹除哈密顿约束算符 $\widehat{H}[N]$ 定义为仅作用于自旋网络的节点，包含一个绕回率算符，该算符会引入对自旋的求和。欧几里得部分的作用可表示为

$$\widehat{H}[N]\,\big|S\rangle = N(x_n)\widehat{H}(x_n)\big|S\rangle = N(x_n)A_n^\alpha(S)\,|S_n^\alpha\rangle \tag{7}$$

where $A_n^\alpha(S)$ are coefficients that can be explicitly computed, summation is understood for the repeated indices n and α , and n runs over the nodes of S and, for each node n, α runs over the choice of all possible pairs of links (ℓ, ℓ') at n as well as over two irreducible representations on each of these links. S_n^α denotes the spin network S with an extra link added connecting ℓ and ℓ' as in Fig. 1, and x_n the spatial position of n . In the original prescription [24], one lets this operator act on diffeomorphism-invariant states, at which point the exact position of the added link does not matter, as long as it is chosen sufficiently close to the node (and then the limit as it approaches the node, corresponding to the removal of the regulator, becomes trivial). For the purpose of deriving the spin foam framework, an exact position for the added link for each spin network and node therein must be chosen, and this must be done in a way that is diffeomorphism covariant, which can always be done, e.g., by making these choices arbitrarily for one spin network in each diffeomorphism equivalence class and then extending these choices to the rest via the action of diffeomorphisms.

其中 $A_n^\alpha(S)$ 是可显式计算的系数，对重复指标 n 和 α 按爱因斯坦求和规则处理，n 遍历 S 的所有节点，对每个节点，n,α 遍历 n 处所有可能的链路对 (ℓ,ℓ')，同时遍历每条链路的所有不可约表示。S_n^α 表示新增一条链路连接 ℓ 和 ℓ' 后的自旋网络 S，如图 1 所示，x_n 为 n 的空间位置。在原始方案 [24] 中，该算子作用在微分同胚不变态上，此时只要新增链路选在足够靠近节点的位置，其精确位置就无关紧要 (当链路趋近节点时，对应移除正则化因子的极限是平凡的)。为推导自旋泡沫框架，必须为每个自旋网络及其内部节点选定新增链路的精确位置，且该选定必须满足微分同胚协变性，这一点总能实现: 例如，对每个微分同胚等价类中的一个自旋网络任意选定位置，再通过微分同胚作用将该选定推广到其余自旋网络即可。

For repeated actions of (7) on a spin network, it is convenient to introduce the notation $S_{n_1\cdots n_m}^{\alpha_1\cdots\alpha_m} := \left(\cdots\left(S_{n_1}^{\alpha_1}\right)\cdots\right)_{n_m}^{\alpha_m}$, so that, for example,

对于 (7) 式在自旋网络上的重复作用，引入记号 $S_{n_1\cdots n_m}^{\alpha_1\cdots\alpha_m} := \left(\cdots\left(S_{n_1}^{\alpha_1}\right)\cdots\right)_{n_m}^{\alpha_m}$ 会更方便，例如

$$\hat{H}[N]^2\left|S\right\rangle = \hat{H}[H]\left(N\left(x_{n_1}\right)A_{n_1}^{\alpha_1}(S)\mid S_{n_1}^{\alpha_1}\rangle\right) = N\left(x_{n_2}\right)N\left(x_{n_1}\right)A_{n_2}^{\alpha_2}(S)A_{n_1}^{\alpha_1}(S)\left|S_{n_1n_2}^{\alpha_1\alpha_2}\right\rangle$$

(8)

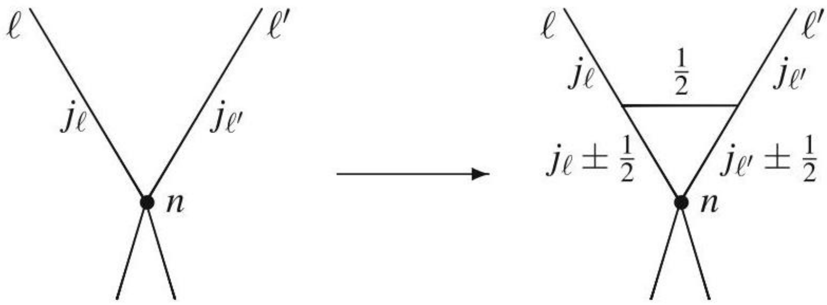

Fig. 1 Action of Thiemann's Hamiltonian constraint operator on a node n of a spin network

图 1 Thiemann 哈密顿约束算符对自旋网络节点 n 的作用

With this notation, continuing the derivation (6),

使用该记号继续推导 (6) 式，得到

$$P\left|s\right\rangle = \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int\mathcal{D}\varphi U(\varphi)\int\mathcal{D}NN\left(x_{n_m}\right)\cdots N\left(x_{n_1}\right)A_{n_m}^{\alpha_m}\left(S_{n_1\cdots n_{m-1}}^{\alpha_1\cdots\alpha_{m-1}}\right)\cdots A_{n_1}^{\alpha_1}(S)\left|S_{n_1\cdots n_m}^{\alpha_1\cdots\alpha_m}\right\rangle$$

$$= \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\left(\int\mathcal{D}NN\left(x_{n_m}\right)\cdots N\left(x_{n_1}\right)\right)A_{n_m}^{\alpha_m}\left(S_{n_1\cdots n_{m-1}}^{\alpha_1\cdots\alpha_{m-1}}\right)\cdots A_{n_1}^{\alpha_1}(S)\left|s_{n_1\cdots n_m}^{\alpha_1\cdots\alpha_m}\right\rangle$$

(9)

where $s^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m} := s\left(S^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m} := \left(\cdots\left(S^{\alpha_1}_{n_1}\right)\cdots\right)^{\alpha_m}_{n_m}\right)$. The diffeomorphism invariance of $\mathcal{D}N$ further ensures that the remaining (formal) integral over the lapse,

其中 $s^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m} := s\left(S^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m} := \left(\cdots\left(S^{\alpha_1}_{n_1}\right)\cdots\right)^{\alpha_m}_{n_m}\right)$。$\mathcal{D}N$ 的微分同胚不变性进一步保证了剩余对推移时间的 (形式) 积分

$$I_{\{m_k\}} := \int \mathcal{D}NN\left(x_{n_m}\right)\cdots N\left(x_{n_1}\right), \tag{10}$$

depends only on the number m_k of points that appear k times for $k = 1, 2, 3, \ldots$. Using the projection (9) to define the physical inner product on solutions to the Hamiltonian constraint, in a manner similar to (5), we have

仅依赖于对 $k = 1, 2, 3, \ldots$ 而言，出现 k 次的点的数量 m_k 。类似式 (5)，我们利用投影 (9) 来定义哈密顿约束解上的物理内积，可得

$$\langle s'|P|s\rangle = \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}I_{\{m_k\}}A^{\alpha_m}_{n_m}\left(S^{\alpha_1\cdots\alpha_{m-1}}_{n_1\cdots n_{m-1}}\right)\cdots A^{\alpha_1}_{n_1}(S)\left\langle s' \mid s^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m}\right\rangle. \tag{11}$$

This expression can be interpreted as a sum over sequences of diffeomorphism-invariant spin networks $\left(s, s^{\alpha_1}_{n_1}, \ldots, s^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m}\right)$ related by consecutive actions of the Hamiltonian constraint, acting as in Fig. 1, and ending in the final spin network s' . Such a sequence of spin networks can be interpreted as a spin foam, a quantum spacetime, matching to the initial and final spin networks s and s' , which have the interpretation of states of quantum "space." Each such spin foam consists in a 2- complex, with each 2-cell, or face f , labeled by a quantum number of area, j_f , and each 1-cell, or edge e , labeled by an intertwiner i_e . An edge branches into multiple edges at a vertex v , corresponding to one of the actions of the Hamiltonian constraint at a single node. These vertices v are thus in one-to-one correspondence with the factors $A^{\alpha_m}_{i_m}\left(S^{\alpha_1\cdots\alpha_{m-1}}_{n_1\cdots n_{m-1}}\right)$ appearing in the above expression. Each such factor depends only on the spins and intertwiners at the node acted upon and the nodes created and, due to the diffeomorphism covariance of $\widehat{H}[N]$, is independent of the choice of representative S averaging to s . We therefore denote each such factor as A_v , the vertex amplitude associated with v . The amplitude (11) can then be written as

该表达式可解释为对微分同胚不变自旋网络序列 $\left(s, s^{\alpha_1}_{n_1}, \ldots, s^{\alpha_1\cdots\alpha_m}_{n_1\cdots n_m}\right)$ 求和，这些自旋网络通过哈密顿约束的连续作用关联，作用方式如图 1 所示，最终终止于末自旋网络 s' 。这样的自旋网络序列可解释为自旋泡沫——即匹配初末自旋网络 s 和 s' 的量子时空，而初末自旋网络本身对应量子“空间”的状态。每个自旋泡沫由一个 2-复形构成: 其中每个 2 单元 (即面 f) 由面积量子数 j_f 标记，每个 1 单元 (即边 e) 由 intertwinner i_e 标记。边在顶点 v 处分支为多条边，对应单个节点上一次哈密顿约束作用。因此这些顶点 v 与上述表达式中出现的因子 $A^{\alpha_m}_{i_m}\left(S^{\alpha_1\cdots\alpha_{m-1}}_{n_1\cdots n_{m-1}}\right)$ 一一对应。每个因子仅依赖于作用节点和新生节点上的自旋与 intertwiner，且由于 $\widehat{H}[N]$ 的微分同胚协变性，它与对 s 做平均时所选的代表元 S 无关。因此我们将每个此类因子记为 A_v ，即关联于 v 的顶点振幅。于是振幅 (11) 可写为

$$\langle s'|P|s\rangle = \sum_{\mathcal{F}|\partial\mathcal{F}=s\cup s'}\frac{(-i)^m}{m!}I_{\{m_k\}}\prod_{v\in\mathcal{F}}A_v(\mathcal{F}), \tag{12}$$

where the sum over $\mathcal{F}$ is a sum over all spin foams with boundary equaling the union of the initial and final spin networks s, s', m is the number of vertices in $\mathcal{F}$, and m_k is the number of nodes in s' with k vertices in $\mathcal{F}$ resulting from the action of $\widehat{H}[N]$ on it.

> 其中对 $\mathcal{F}$ 的求和是对所有边界等于初末自旋网络并集的自旋泡沫求和, s, s', m 是 $\mathcal{F}$ 中的顶点数, m_k 是 s' 中的节点数，其中 $\mathcal{F}$ 内的 k 顶点由 $\widehat{H}[N]$ 作用于该节点产生。

The resulting expression is however still unsatisfactory. The vertex amplitude $A_v(\mathcal{F})$ depends on how the spin foam $\mathcal{F}$ is interpreted as a history of spin networks, which can be related to a choice of foliation of spacetime. Furthermore, the summand, due to the factor $I_{\{m_k\}}$, is not local in time. In addition, it is not yet clear how to make sense of the factors $I_{\{m_k\}}$ resulting from a functional integral over lapses [23]. Therefore, even though the construction of a well-defined Hamiltonian operator was a key feature to support the viability of the program, there are unfortunately ambiguities in its definition, and it is not even clear that it satisfies all necessary physical requirements. The idea to overcome these limitations is to take the above expression as a suggestion for a more general ansatz, which keeps the key state sum structure and makes foliation independence manifest:

> 但由此得到的表达式仍不尽人意。顶点振幅 $A_v(\mathcal{F})$ 依赖于将自旋泡沫 $\mathcal{F}$ 解释为自旋网络演化史的方式，这与时空叶状结构的选择相关。此外，由于因子 $I_{\{m_k\}}$，求和项不具有时间局域性。同时，目前还不清楚如何理解 lapse 泛函积分得到的因子 $I_{\{m_k\}}$ [23]。因此，尽管构造良定义的哈密顿算符是支撑该方案可行性的核心特征，但遗憾的是它的定义存在歧义，甚至尚且不清楚它是否满足所有必要的物理要求。克服这些局限的思路是，将上述表达式作为更广义近似解的参考，该近似保留核心的态和结构，并明确体现叶状结构无关性:

$$K[s' \cup s] := \langle s' | P | s \rangle = \sum_{\mathcal{F} | \partial \mathcal{F} = s \cup s'} A_{\mathcal{F}}, \; A_{\mathcal{F}} = \prod_f A_f \prod_e A_e \prod_v A_v,$$

(13)

where each A_f depends only on the spin label j_f on f, A_e depends only on the intertwiner i_e on e , and A_v depends only on the spins and intertwiners on the faces and edges incident at the vertex v . The goal is to find expressions for the weights such that the quantum constraints are correctly implemented and general relativity is recovered in the appropriate limit.

> 其中每个 A_f 仅依赖于自旋标记 j_f ， f, A_e 仅依赖于缠结 i_e 在 e 上，A_v 仅依赖于顶点 v 关联面和边上的自旋与缠结。我们的目标是找到权重的表达式，使得量子约束被正确满足，且广义相对论能在合适的极限下还原。

Further support for a formula like (13) comes from the fact that different and independent approaches have converged to it, suggesting that an expression of this type may be seen as a definition of a generally covariant QFT [2]. State sum models for topological quantum field theories, such as the Ooguri and Crane-Yetter models for BF theory [25,26] and the Ponzano-Regge and Turaev-Viro models for 3D gravity [27, 28], take this form, as do group field theories, appropriately cast [29]. It is referred to as the spin foam ansatz, and it forms the basis of the spin foam framework for defining dynamics for loop quantum gravity. Requiring the composition property

式 (13) 得到进一步支持的原因在于，多个独立的不同研究方向都得到了这一结果，说明这类表达式可以被视为广义协变量子场论的定义 [2]。拓扑量子场论的态和模型，例如 BF 理论的 Ooguri 模型和 Crane-Yetter 模型 [25,26]，以及三维引力的 Ponzano-Regge 模型和 Turaev-Viro 模型 [27,28]，都采用了这种形式，经恰当整理后的群场论也是如此 [29]。这被称为自旋泡沫近似，它是圈量子引力定义动力学的自旋泡沫框架的基础。对复合性质的要求

$$\sum_{s''} K\left[s' \cup s''\right] K\left[s'' \cup s\right] = \sum_{s''} \langle s' \left| P \right| s'' \rangle \langle s'' \left| P \right| s \rangle = \langle s' \left| P \right| s \rangle = K\left[s' \cup s\right]$$

(14)

fixes the face amplitudes to be $A_f = 2j_f + 1$ [30]. Furthermore, the edge amplitude can always be absorbed into the vertex amplitude, so that the latter carries all the dynamical information of the model.

将面振幅固定为 $A_f = 2j_f + 1$ [30]。此外，边振幅总可以被吸收进顶点振幅，因此顶点振幅承载了模型的全部动力学信息。

The formula (13) has the same structure as a partition function for lattice gauge theory (LGT), and some techniques for its study are indeed common to the two fields. However, there are also crucial differences stemming from the differences between quantizing a gauge theory on Minkowski spacetime and quantizing general relativity with no background structures. In particular, LGT can use a regular lattice adapted to flat spacetime, and this is pivotal in order to derive the right universal class of actions, define observables and length scales, compute the continuum limit, and verify the existence of a phase transition. Conversely, background independence means that one can only use abstract graphs, whose regularity and locality are a priori distinct from that of any emerging semiclassical spacetime, so that verifying that the action is related to gravitational dynamics becomes subtle and non-trivial, and there is as yet no consensus on a procedure to define the relevant scales and corresponding choice of boundary and bulk structure to compute specific physical processes. Similarly, the refinement limit and how to sum over refined contributions are open questions, whose study is the object of ongoing research. See in particular the Chap. 93, ”Spin Foams, Refinement Limit, and Renormalization” in this book dedicated to these questions. As we will comment on in the conclusions, more explicit results are needed before one can discern between different procedures put forward in the literature. On a more technical level, the LGT dynamics is encoded in the face amplitude via the plaquette action, and the vertex amplitude carries only information about gauge invariance, whereas in most spin foam models, it is the reverse: It is the vertex amplitude that encodes dynamics, and the face amplitude is determined by gauge invariance with no dynamical information. The similarities and differences between LGT and spin foam quantum gravity are a rich topic for discussions; see, e.g., [31].

式 (13) 与格点规范理论 (LGT) 的配分函数结构相同，因此二者确实共享不少研究技巧。但由于闵氏时空上规范理论量子化与无背景结构的广义相对论量子化之间存在差异，二者也有关键区别。具体而言，格点规范理论可以使用适配平直时空的规则格点，这对于得到正确的普适作用量类、定义可观测量和长度标度、计算连续极限、验证相变存在性都至关重要。反之，背景独立性意味着我们只能使用抽象图，抽象图的规则性和局域性先天地不同于任何涌现半经典时空的规则性和局域性，因此验证作用量与引力动力学相关变得十分微妙且不平凡，目前对于定义相关标度、对应选择边界与体结构来计算具体物理过程的步骤，学界尚未达成共识。同样，精化极限以及如何对精化贡献求和都是开放性问题，目前仍是 ongoing 研究的课题，相关研究详见本书第 93 章《自旋泡沫、精化极限与重整化》。正如我们在结论中将要讨论的，在区分文献中提出的不同方案之前，还需要更多明确的结果。从更技术的层面来说，格点规范理论的动力学通过 plaquette 作用编码在面振幅中，顶点振幅仅携带规范不变性的信息；而在大多数自旋泡沫模型中情况正好相反: 动力学由顶点振幅编码，面振幅由规范不变性决定，不携带动力学信息。格点规范理论和自旋泡沫量子引力的异同是一个值得深入探讨的课题，相关讨论参见例如 [31]。

Spin Foam Amplitudes from the Plebanski Action

来自普莱班斯基作用量的旋泡沫振幅

As noted, an expression like (13) is known to arise in quantum BF theories; furthermore, in 4d, the action for general relativity can be written as a constrained BF action. This suggests the following approach: first quantize BF theory, thus obtaining an amplitude already in the form (13), and then impose constraints reducing BF to gravity directly at the quantum level, as a restriction on the quantum states allowed. There is no guarantee that this procedure gives the same result as quantizing the constrained theory, if one knew how to do that. The logic used is that one will have to verify a posteriori whether one obtains in this way a theory with the right semiclassical dynamics and capable of making new predictions. The value of this approach is that one obtains in a natural way both a background-independent quantum path integral and boundary states given by SU(2) spin networks. There exist excellent overviews of this approach, e.g., [32]. We refer the reader there for details and only sketch here the steps necessary to follow the rest of the chapter.

如前所述，(13) 这样的表达式已知出现在量子 BF 理论中；此外，在四维情况下，广义相对论的作用量可以写为带约束的 BF 作用量。这提示了如下研究思路: 先对 BF 理论量子化，得到已经具有 (13) 形式的振幅，再直接在量子层面施加约束，将 BF 理论约化为引力，作为对允许量子态的限制。没人能保证这个过程得到的结果会和直接量子化带约束理论 (如果我们知道怎么量子化的话) 的结果一致。我们采用的逻辑是，需要后验验证: 通过这种方式得到的理论是否具备正确的半经典动力学，并且能够给出新预言。该方法的优势在于，它能自然得到背景无关的量子路径积分，以及由 SU(2) 自旋网络给出的边界态。关于该方法已有出色的综述，例如文献 [32]。我们建议读者参阅综述了解细节，本文仅概述理解本章后续内容所需的步骤。

The canonical structure of LQG can be derived from the tetrad action for general relativity. An equivalent way of studying the same theory is to use a version of the Plebanski action [33-35]. The version used is often called covariant or non-chiral, to distinguish it from the original Plebanski action, which is identical but uses self-dual variables only, and has to be supplemented with reality conditions. It is given by

圈量子引力的正则结构可以从广义相对论的标架作用量导出。研究该理论的另一种等价方式是使用普莱班斯基作用量的一种形式 [33-35]。目前常用的这种形式通常被称为协变形式或非手征形式，以此区别于原始的普莱班斯基作用量——后者形式相同，但仅使用自对偶变量，还需要补充实条件。常用形式可写为:

$$S\left(B^{IJ},\omega^{IJ},\phi_{IJKL}\right)=\frac{1}{16\pi G}\int\operatorname{tr}\left(P_\gamma B\wedge F(\omega)-\frac{1}{2}\left(\phi+\frac{\Lambda}{3}P_\gamma\right)B\wedge B\right),$$

(15)

where ω^{IJ} is an $SO(3,1)$ connection with curvature F, B^{IJ} an auxiliary 2-form valued in the adjoint, and

其中 ω^{IJ} 是曲率为 F, B^{IJ} 的 $SO(3,1)$ 联络，F, B^{IJ} 是伴随表示取值的辅助 2 形式，且

$$P_\gamma=\frac{\mathbb{1}}{\gamma}+\star,\ P_\gamma^{-1}=\frac{\gamma}{1+\gamma^2}\left(\mathbb{1}-\gamma\star\right). \tag{16}$$

The trace, tr, here means contracting all indices with the identity $\mathbb{1}_{IJKL}:=\eta_{I[K}\eta_{L]J}$ and $\star:=\frac{1}{2}\varepsilon_{IJKL}$. The field ϕ^{IJKL} is a Lagrange multiplier with the same symmetries as the Riemann tensor and hence has 20 independent components. Its variation gives the following equations:

此处迹 tr 指用恒等式 $\mathbb{1}_{IJKL}:=\eta_{I[K}\eta_{L]J}$ 和 $\star:=\frac{1}{2}\varepsilon_{IJKL}$ 收缩所有指标。场 ϕ^{IJKL} 是拉格朗日乘子，具有和黎曼张量相同的对称性，因此有 20 个独立分量。对其变分可得如下方程:

$$B^{IJ}\wedge B^{KL}=\frac{1}{12}\varepsilon^{IJKL}\operatorname{tr}\left(B\wedge\star B\right). \tag{17}$$

Solutions satisfying the non-degeneracy condition $\operatorname{tr}(B\wedge\star B)\neq 0$ are parametrized by the existence of a non-degenerate tetrad e^I_μ such that one of the following four forms holds [34-36]:

满足非退化条件 $\operatorname{tr}(B\wedge\star B)\neq 0$ 的解由存在非退化标架 e^I_μ 参数化，满足该条件的解可取以下四种形式之一 [34-36]:

$$(I\pm)\ B^{IJ}=\pm e^I\wedge e^J,\ (II\pm)\ B^{IJ}=\pm\star e^I\wedge e^J. \tag{18}$$

A 2-form written as the wedge product of two 1-forms is called simple, and (17) are called (covariant) simplicity constraints. Plugging in the form $(I\pm)$ recovers the original tetrad formulation with Newton constant $\pm G$ and Barbero-Immirzi parameter γ. Plugging in the form $(II\pm)$ recovers the formulation with Newton constant $\pm\gamma G$ and Barbero-Immirzi parameter $-1/\gamma$. At the classical level, there is no problem in choosing either sector for any finite value of γ, provided the right Newton constant is identified. At the quantum level however, these two sectors could lead to non-trivial interference. The linear reformulation of the simplicity constraints used in the EPRL model, which we introduce below (19), fortunately eliminates the sectors $(II\pm)$.

可写为两个 1 形式外积的 2 形式称为单 2 形式，(17) 因此被称为 (协变) 单性约束。代入形式 ($I\pm$) 即可得到原始标架表述，其中牛顿常数为 $\pm G$，巴贝罗-伊米尔齐参数为 γ。代入形式 ($II\pm$) 得到的表述中，牛顿常数为 $\pm\gamma G$，巴贝罗-伊米尔齐参数为 $-1/\gamma$。经典层面上，只要对应正确的牛顿常数，对 γ 的任意有限值选择任意一个扇区都没有问题。但在量子层面，这两个扇区可能会产生非平凡干涉。我们后文 (19) 处引入的 EPRL 模型使用了单性约束的线性重构，幸运的是该构造消除了 ($II\pm$) 扇区。

The covariant Plebanski action can also be seen as a theory for the two Urbantke metrics corresponding to self-dual and anti-self-dual sectors and the simplicity constraints as imposing the matching of the two metrics [37, 38]. This approach is relevant for recent proposals to construct effective spin foam theories [39].

协变普莱班斯基作用量也可以视作对应自对偶扇区和反自对偶扇区的两个厄本班克度量的理论，单性约束的作用是要求两个度量匹配 [37, 38]。该思路与近期构造有效旋泡沫理论的方案相关 [39]。

The canonical structure of the theory is the following. After 3+1 splitting, the phase space is described by the conjugated pair $(\omega_a^{IJ}, \tilde{P}^a_{IJ})$ and has 36 dimensions per space point. Here, $\tilde{P}^{aIJ} := \frac{1}{2}\tilde{\varepsilon}^{abc}J^{IJ}_{bc}$, and $J := P_\gamma B/16\pi G$, and the index $a = 1,2,3$ is over each leaf of the foliation. The momenta are not all independent and have to satisfy six primary simplicity constraints which are the canonical version of the quadratic covariant constraints (17). These are not stable under evolution and give rise to six secondary simplicity constraints, which coincide with the $D_{[a}e^0{}_{b]} = 0$ plus three of the $D_{[a}e^i{}_{b]} = 0$ torsionless equations (the remaining six give the Gauss constraints). Together, the primary and secondary simplicity constraints form a second-class system. This together with the six Gauss constraints and the four diffeomorphism constraints reduces the phase space to four dimensions per space point, namely, the theory has two degrees of freedom per point.

该理论的正则结构如下。进行 3+1 分解后，相空间由共轭对 $(\omega_a^{IJ}, \tilde{P}^a_{IJ})$ 描述，每个空间点具有 36 个维度。其中 $\tilde{P}^{aIJ} := \frac{1}{2}\tilde{\varepsilon}^{abc}J^{IJ}_{bc}$、$J := P_\gamma B/16\pi G$，指标 $a = 1,2,3$ 遍历叶状结构的每一个叶。动量并非全部独立，必须满足六个初阶简单性约束，这是二次协变约束 (17) 的正则形式。这些约束在演化下不具有稳定性，会衍生出六个二阶简单性约束，它们与 $D_{[a}e^0{}_{b]} = 0$ 加上三个 $D_{[a}e^i{}_{b]} = 0$ 无挠方程重合 (其余六个给出高斯约束)。初阶与二阶简单性约束共同构成一个第二类系统。结合六个高斯约束和四个微分同胚约束，最终将每个空间点的相空间约化为四维，即该理论每个点存在两个自由度。

It is often convenient to work fixing one of the tetrad forms, say e^0, in terms of the hypersurface normal. This breaks part of the internal gauge symmetry and introduces three additional constraints which form a second-class pair with the generators of the broken internal symmetry (namely the boosts in the standard case of a space-like 3+1 splitting; working with this partial gauge fixing is also convenient if one wants to describe time-like or null foliations [40]). The three new constraints are linear [41-43],

固定其中一种标架形式 (例如固定为超曲面法向方向的 e^0) 进行计算通常会更方便。这会破缺部分内规范对称性，并引入三个额外约束，它们与被破缺的内对称性生成元构成第二类对 (在类空 3+1 分解的标准情形下就是 boost；如果要描述类时或类空叶状结构，这种部分规范固定也很方便 [40])。这三个新约束是线性的 [41-43],

$$N_I\,(\mathbb{1} - \gamma\star)\,J^{IJ}_{ab} = 0 \tag{19}$$

and imply, and so can replace, the primary simplicity constraints (17). These are nine independent constraints per point. The counting now has $9 + 9$ second-class and $3 + 4$ first-class constraints, giving 4 dimensions per space point as before.

它们蕴含 (因此也可以替代) 初阶简单性约束 (17)。每个点共有九个独立约束。计数结果为 9+9 个第二类约束和 3+4 个第一类约束，和之前一样每个空间点对应四维相空间。

Let us neglect for a moment the spatial indices ab (imagine, for instance, integrating the 2-form on a given surface). If we choose the canonical time direction $N^I = t^I := (1,0,0,0)$, this equation is equivalent to

我们暂时忽略空间指标 ab (例如可以想象对给定曲面上的 2 形式做积分)。如果我们选取正则时间方向 $N^I = t^I := (1,0,0,0)$ ，该方程等价于

$$\mathbf{K} = \gamma \mathbf{L}, \tag{20}$$

where $K^i := J^{0i}, L_i := -\frac{1}{2}\varepsilon_{ijk}J^{jk}$. For an arbitrary direction $N^I := \Lambda^I{}_J t^J$, we have instead [44]

其中 $K^i := J^{0i}, L_i := -\frac{1}{2}\varepsilon_{ijk}J^{jk}$ 。而对于任意方向 $N^I := \Lambda^I{}_J t^J$ ，我们得到的结果是 [44]

$$\cosh\eta\,(\mathbf{K} - \gamma\mathbf{L}) = \sinh\eta\left(R^{-1}\mathbf{u}\right) \times (\mathbf{L} + \gamma\mathbf{K}), \tag{21}$$

where the Lorentz transformation Λ has been decomposed as a product of a rotation R times a boost with rapidity η and direction u . While B is simple, we refer to J satisfying (20) as γ -simple and (21) as boosted γ -simple.

其中洛伦兹变换 Λ 被分解为旋转 R 乘以快度为 η 、方向为 u 的 boost。当 B 是简单的，我们将满足 (20) 的 J 称为 γ -简单，将 (21) 称为 boosted γ -简单。

The relevance of this action for us is that it formulates general relativity as a topological field theory plus constraints. A topological field theory can be quantized without any reference to a background metric structure, in two steps: first, invoking topological invariance to restrict the partition function/generating functional to a cellular decomposition and second, choosing to represent it as a product of delta functions imposing flatness of the holonomies on the faces of this decomposition. Gauge divergences are treated by removing redundant delta functions. The result is finite and can be naturally recast as a state sum model using the character expansion of the deltas. Integrating out gives precisely an expression like (13), with the amplitudes obtained from the recoupling theory of the chosen gauge group. This quantization of BF theory will be reviewed in details in the next section.

该作用量对我们的意义在于，它将广义相对论表述为拓扑场论加上约束。拓扑场论可以不依赖背景度规结构完成量子化，分为两步: 第一，利用拓扑不变性将配分函数/生成泛函限制到胞分解上；第二，将其表示为 δ 函数的乘积，这些 δ 函数会限制全曲率在分解的面上是平坦的。规范发散可以通过移除冗余 δ 函数处理。得到的结果是有限的，利用 δ 函数的特征展开可以自然地将其重构为状态和模型。积分后恰好得到形如 (13) 的表达式，其中振幅由所选规范群的重耦理论得到。BF 理论的这种量子化将在下一节详细回顾。

It then leads to the following strategy to construct spin foam models: start from the quantum BF theory and impose the primary simplicity constraints, which have a simple action and can be done with a restriction on the quantum labels of the state sum model. This restriction destroys the proof of topological invariance; therefore, the resulting model could carry local degrees of freedom in the continuum limit. Furthermore, the limit of large label is related to the Regge action, which offer a possibility toward proving that the continuum limit is related to GR.

由此它引出了构建旋子泡沫模型的下述策略: 从量子 BF 理论出发，施加原初简单性约束，该约束作用形式简单，只需对态和模型的量子标记做限制即可实现。这一限制破坏了拓扑不变性的证明；因此，最终得到的模型在连续极限下可以承载局域自由度。此外，大标记极限与雷杰作用量相关，这为证明连续极限与广义相对论相关提供了可能。

Different ways of implementing this strategy have been considered in the literature. One has to first choose a discretization of the classical constraints adapted to the cellular decomposition used and then choose a quantization map. The simplest choice of cellular decomposition is a simplicial one: In this case, a piece-wise flat geometry can be unambiguously described by the edge lengths, and this description forms the basis of Regge calculus. The more adaptable choice is to work with all tetrahedra being space-like. This is the case most used in the literature. We will comment in section "Extension to Non-space-Like Building Blocks" on other choices. On each 4-simplex, the quadratic covariant constraints (17) can be discretized integrating the B field on different triangles of the 4-simplex. The resulting constraints can be split into three groups [45], diagonal, cross-diagonal, and opposite-faces. Gauge invariance can be implemented in the form of closure constraints. Flatness of the 4-simplex and closures then imply that the opposite-faces constraints are redundant and can be discarded. Indeed, the diagonal and cross-diagonal together with the closure constraints are enough to guarantee that the bivectors discretizing the B field are in one-to-one correspondence with an oriented Regge 4-simplex. Since the opposite-faces constraints are the only ones to involve the connection, these two results could be taken as a signal that imposing the primary constraints at all times, one is effectively inducing the secondary ones as well. This property, however, is lost if one uses a more general cellular decomposition, because in this case the opposite-faces constraints become non-trivial, and furthermore, there is no unique map between edge lengths and piece-wise flat metrics. Finally, the closure constraints also guarantee that the diagonal and cross-diagonal are solved when linear simplicity is solved [46]. Hence, one can trade those quadratic constraints for the linear ones, with the advantage of removing already at the classical level the presence of the sectors $(II\pm)$ (18) .

文献中已经讨论过实现该策略的不同方式。研究者需要首先选择一个适配所用胞分解的经典约束离散化方案，再选择量子化映射。胞分解最简单的选择是单纯形分解: 在这种情况下，分段平坦几何可以由边长唯一描述，该描述构成了雷杰微积分的基础。适应性更强的选择是令所有四面体都是类空的。这是文献中最常用的情况。我们会在"非类空构造块的拓展"一节讨论其他选择。在每个 4 单纯形上，二次协变约束 (17) 可以通过对 4 单纯形的不同三角形积分 B 场完成离散化。得到的约束可以分为三组 [45]: 对角约束、交叉对角约束和对面约束。规范不变性可以表示为闭合约束的形式。4 单纯形的平坦性加上闭合性意味着对面约束是冗余的，可以丢弃。事实上，对角约束、交叉对角约束加上闭合约束就足以保证离散化 B 场的二向量与定向雷杰 4 单纯形一一对应。由于只有对面约束涉及联络，这两个结论可以说明: 始终施加原初约束实际上也会自然导出次级约束。但如果使用更一般的胞分解，这一性质就会消失，因为此时对面约束变为非平凡约束，此外边长和分段平坦度量之间不存在唯一映射。最后，闭合约束还可以保证，当线性简单性约束得到求解时，对角约束和交叉对角约束也一并得到求解 [46]。因此，我们可以用线性约束替换这些二次约束，其优势是在经典层面就可以消除扇区 $(II\pm)$ (18) 的存在。

Within this classical set-up, two different quantization maps have been studied. The first one was the Barrett-Crane (BC) model [45,47-49], later followed by the Engle-Pereira-Rovelli-Livine (EPRL) model [41,42,46,50,51]. We choose to focus here on the EPRL model for two reasons. First, it is currently favored because it overcame difficulties with correlators [52], implements weakly the non-commuting constraints [41,51], can accommodate an arbitrary Immirzi parameter [42,50], and can be easily extended to arbitrary graphs [53,54]. Second, it is sufficiently general and versatile that it can be used as a starting point to then learn the simpler BC model or the more complicated models with building blocks of different spacetime signatures. In the next section, we present in some details the EPRL model. For other work on the BC model, see, e.g., [55-59].

在这个经典框架下, 已经有两种不同的量子化映射得到研究。第一种是巴雷特-克兰 (BC) 模型 [45,47-49]，后续出现了恩格尔-佩雷拉-罗韦利-利文 (EPRL) 模型 [41,42,46,50,51]。我们在这里选择聚焦 EPRL 模型，原因有二。首先，它目前更受认可: 它解决了关联函数的相关困难 [52]，弱实现了非对易约束 [41,51]，可以容纳任意伊米尔齐参数 [42,50]，还可以轻易推广到任意图 [53,54]。其次，它足够通用灵活，可以作为学习更简单的 BC 模型，或是构造块对应不同时空符号的更复杂模型的起点。在下一节，我们将较为详细地介绍 EPRL 模型。关于 BC 模型的其他研究，参见例如 [55-59]。

EPRL Model

EPRL 模型

The EPRL model is based on the Lie group $\mathrm{SL}(2,\mathbb{C})$, which describes the local Lorentz symmetry of general relativity. In this section, we briefly review relevant aspects of this group and explain how they are used to define the EPRL model and its relation to topological BF theory. We then review the key properties of the model, starting from its large spin limit, as well as various extensions thereof that have been studied.

EPRL 模型基于李群 $\mathrm{SL}(2,\mathbb{C})$ ，该李群描述了广义相对论的局域洛伦兹对称性。本节我们简要回顾该群的相关性质，说明如何利用这些性质定义 EPRL 模型，以及该模型与拓扑 BF 理论的关系。随后我们从大自旋极限开始，回顾该模型的核心性质，以及目前已被研究过的该模型的各类扩展形式。

Unitary Irreps of SL(2, C)

SL(2, C) 的幺正不可约表示

The EPRL model is based on the principal series of the unitary irreducible representations (irreps) of $\mathrm{SL}(2,\mathbb{C})$, which are infinite dimensional since the group is non-compact. These can be labeled by a pair $(\rho,k)\in(\mathbb{R},\mathbb{N}/2)$, with Casimirs

EPRL 模型基于 $\mathrm{SL}(2,\mathbb{C})$ 的酉不可约表示 (不可约表示) 的主 Series，由于该群是非紧致群，这些表示是无穷维的。它们可以由一对标记 $(\rho,k)\in(\mathbb{R},\mathbb{N}/2)$ 标识，对应卡西米尔算子为

$$\hat{C}_1 := \hat{\mathbf{L}}^2-\hat{\mathbf{K}}^2=\left(k^2-\rho^2-1\right)\mathbb{1},\ \hat{C}_2 := \hat{\mathbf{K}}\cdot\hat{\mathbf{L}}=\rho k\mathbb{1}. \tag{22}$$

Here, $\left(\hat{\mathbf{L}},\hat{\mathbf{K}}\right)$ are the rotation and boost generators, as defined by the canonical time direction $t^I := (1,0,0,0)$. A basis for these irreps can be obtained looking at representations of any little group. For the EPRL model, it is convenient to work with Naimark's canonical orthonormal basis, labeled by the SU(2) little group with $\hat{\mathbf{L}}^2$ and $\hat{L}_z$: $|\rho,k;j,m\rangle$, where $j\geq k$ and $m\in[-j,j]$ is the magnetic number. The matrix elements in this basis are $D^{(\rho,k)}_{jmln}(h)$, where jm and ln refer, respectively, to the right-invariant and left-invariant realizations of the little group. If $h=g\in\mathrm{SU}(2)$, then $D^{(\rho,k)}_{jmln}(g)=\delta^{jl}D^{(j)}_{mn}(g)$, for all ρ and $k\leq j$, where $D^{(j)}_{mn}(g)$ are the usual Wigner D-matrices for $\mathrm{SU}(2)$.

此处，$\left(\hat{\mathbf{L}},\hat{\mathbf{K}}\right)$ 是由标准时间方向 $t^I := (1,0,0,0)$ 定义的旋转生成元和 boost 生成元。我们可以通过研究任意迷向群的表示得到这些不可约表示的一组基。对于 EPRL 模型，使用奈马克标准正交基十分方便，该基由带有 $\hat{\mathbf{L}}^2$ 和 $\hat{L}_z$: $|\rho,k;j,m\rangle$ 的 SU(2) 迷向群标记，其中 $j\geq k$ ，$m\in[-j,j]$ 是磁量子数。该基下的矩阵元为 $D^{(\rho,k)}_{jmln}(h)$ ，其中 jm 和 ln 分别对应迷向群的右不变实现和左不变实现。若 $h=g\in\mathrm{SU}(2)$ ，则对所有 ρ 和 $k\leq j$ 都有 $D^{(\rho,k)}_{jmln}(g)=\delta^{jl}D^{(j)}_{mn}(g)$ ，其中 $D^{(j)}_{mn}(g)$ 是 $\mathrm{SU}(2)$ 对应的标准维格纳 D 矩阵。

The label m of the orthonormal basis can be given a geometric interpretation as projection along the $\hat{z}$ axis of the rotation generator $\hat{L}^i$. As such, each state has minimal information about the direction of the rotation generator. A sharper characterization can be obtained working with $\mathrm{SU}(2)$ coherent states, which can be embedded in the unitary irreps of $\mathrm{SL}(2,\mathbb{C})$ as follows:

标准正交基的标记 m 可以给出几何解释: 它是旋转生成元 $\hat{L}^i$ 沿 $\hat{z}$ 轴的投影。因此，每个状态都携带旋转生成元方向的最小信息。使用 $\mathrm{SU}(2)$ 相干态可以得到更清晰的刻画，$\mathrm{SU}(2)$ 相干态可以按如下方式嵌入 $\mathrm{SL}(2,\mathbb{C})$ 的酉不可约表示中:

$$\left|\rho,k;j,\zeta\right\rangle := D^{(j)}(\zeta)\left|\rho,k;j,-j\right\rangle. \tag{23}$$

Here, $D(\zeta)$ is the Hopf section, and the magnetic index is replaced by a complex number ζ which represents a point on the sphere via stereographic projection. We will denote $n(\zeta)$ the corresponding unit vector in $\mathbb{R}^3$. Note this $\mathrm{SL}(2,\mathbb{C})$ embedding of $\mathrm{SU}(2)$ coherent states should not be confused with the $\mathrm{SL}(2,\mathbb{C})$ coherent states defined in [60], which exist only for $k=0$. The expectation values of rotations and boosts point in the direction identified by the state

此处，$D(\zeta)$ 是霍普夫截面，磁指标由复数 ζ 替换，该复数通过球极投影表示球面上的一点。我们将 $\mathbb{R}^3$ 中对应的单位向量记为 $n(\zeta)$ 。请注意，不要将这种 SL (2, ℂ) 嵌入下的 SU (2) 相干态与文献 [60] 中定义的 SL (2, ℂ) 相干态混淆，后者仅存在于 $k = 0$ 。旋转和 boost 的期望值指向该态确定的方向

$$\mathbf{L} := \langle \rho, k; j, \zeta \left| \hat{\mathbf{L}} \right| \rho, k; j, \zeta \rangle = -jn(\zeta), \tag{24a}$$

$$\mathbf{K} := \langle \rho, k; j, \zeta \left| \hat{\mathbf{K}} \right| \rho, k; j, \zeta \rangle = -\frac{\rho k}{j+1} n(\zeta). \tag{24b}$$

The uncertainty is minimized in the direction of the rotation generators - but not of the boost generators. From this relation, it follows that

旋转生成元方向的不确定性被最小化，但 boost 生成元方向的不确定性没有。由此关系可以推出

$$\mathbf{K} - \frac{\rho k}{j(j+1)} \mathbf{L} = 0. \tag{25}$$

Classical and Quantum BF Theory in Four Dimensions

四维经典与量子 BF 理论

The basic variables of SL (2, ℂ) BF theory on a four-dimensional manifold M are an SL (2, ℂ) connection ω^{IJ} and two-form J^{IJ} taking values in the Lie algebra $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$, and the action is given by

四维流形 M 上 SL (2, ℂ) BF 理论的基本变量是取值于李代数 $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$ 的 SL (2, ℂ) 联络 ω^{IJ} 和二元型 J^{IJ} ，其作用量为

$$S_{BF}(\omega, J) = \int_M \operatorname{tr}(J \wedge F) \tag{26}$$

where F^{IJ} is the curvature of ω^{IJ} . Variation gives

其中 F^{IJ} 是 ω^{IJ} 的曲率。变分可得

$$\delta S_{BF}(\omega, J) = \int_M \operatorname{tr}(\delta J \wedge F - (d_\omega J) \wedge \delta\omega) + \int_{\partial M} \operatorname{tr}(J \wedge \omega) \tag{27}$$

from which one can read off the equations of motion

由此可以直接读出运动方程

$$F \approx 0 \ \text{(flatness)}, \tag{28}$$

$$d_\omega J \approx 0 \ \text{(covariantly constant } J \text{)}, \tag{29}$$

and symplectic potential $\Theta(\delta) = \int_{\partial M} J \wedge \delta\omega$, yielding a boundary phase space parameterized by the pull-backs $(\underline{J}, \underline{\omega})$ of the basic variables to the boundary, with Poisson brackets

以及辛势 $\Theta(\delta) = \int_{\partial M} J \wedge \delta\omega$ ，得到由基本变量拉回至边界的 $(\underline{J}, \underline{\omega})$ 参数化的边界相空间，其泊松括号为

$$\left\{J_{ab}^{IJ}(x), \omega_c^{KL}(y)\right\} = \varepsilon_{abc} \mathbb{1}^{IJKL} \delta^{(3)}(x, y) \tag{30}$$

with ε_{abc} the Levi-Civita symbol of density weight -1 on ∂M . Holding ω constant on the boundary, the (formal) continuum path integral is then

其中 ε_{abc} 是 ∂M 上密度权为-1 的列维-奇维塔符号。固定边界上的 ω ，(形式化的) 连续路径积分即为

$$W\left(\underset{\leftarrow}{\omega}\right) = \int \mathcal{D}\omega \mathcal{D}J e^{i\int_M \operatorname{tr}(J \wedge F)} = \int \mathcal{D}\omega \prod_{x \in M} \delta(F) \tag{31}$$

which manifestly implements the flatness constraint (28). Introduce an oriented cell complex $\mathcal{K}$ covering $M.\omega$ is naturally discretized by the parallel transport $h_e \in \mathrm{SL}(2, \mathbb{C})$ it defines along each interior 1-cell, that is, edge, e in $\mathcal{K}$. The edges in the boundary $\partial\mathcal{K}$ we call links ℓ , which together form the boundary graph γ . The corresponding parallel transports h_ℓ provide the natural discretization of ω . The (formal) continuum path integral (31) then has the obvious discretization

它显然满足平坦性约束 (28)。引入覆盖 $M.\omega$ 的定向胞腔复形 $\mathcal{K}$ ，可通过它在 $\mathcal{K}$ 中每个内部 1 胞腔 (即边) e 上定义的平行移动 $h_e \in \mathrm{SL}(2, \mathbb{C})$ 自然完成离散化。我们将边界 $\partial\mathcal{K}$ 中的 1 胞腔称为链接 ℓ ，它们共同构成边界图 γ 。对应的平行移动 h_ℓ 就是 ω 的自然离散化。那么 (形式化的) 连续路径积分 (31) 的直观离散化形式为

$$W(\{h_\ell\}) = \int \left(\prod_{e \in \mathcal{K}} dh_e\right) \prod_{f \in \mathcal{K}} \delta\left(\overrightarrow{\prod} e \in f h_e\right), \tag{32}$$

where the product over f is the product over all interior 2-cells, or faces, in $\mathcal{K}$ and the product inside of the delta function is over the 1-cells in the boundary of f , whether interior or boundary (link). dh_e denotes the Haar measure, and $\delta(\cdot)$ denotes the Dirac delta with respect to this measure, peaked at the identity.

其中 f 的乘积是对 $\mathcal{K}$ 中所有内部 2 胞腔 (即面) 求乘积，德尔塔函数内部的乘积是对 f 边界上的 1 胞腔 (无论是内部还是边界链接) 求乘积。dh_e 表示哈尔测度，$\delta(\cdot)$ 表示该测度下峰值在单位元处的狄拉克德尔塔函数。

The group elements h_ℓ associated with the boundary graph γ parameterize the discrete boundary configuration space $\mathcal{C}$, so that $\mathcal{C} \cong \mathrm{SL}(2, \mathbb{C})^L$, where L is the number of links. To discretize the conjugate variable J , for each link ℓ of γ , introduce an oriented 2-surface S_ℓ containing the "source" node ℓ_- of ℓ , with ℓ "above" S_ℓ and all other links at ℓ_- "below" S_ℓ , and define the flux $J_\ell^{IJ} := \int_{S_\ell} J^{IJ} \in \mathfrak{so}(3,1) \cong \mathfrak{sl}(2, \mathbb{C})$. Then (30) yields the Poisson brackets [61]

与边界图 γ 关联的群元 h_ℓ 参数化了离散边界构型空间 $\mathcal{C}$, 因此有 $\mathcal{C} \cong \mathrm{SL}(2,\mathbb{C})^L$, 其中 L 是链接的数量。为了离散化共轭变量 J , 对 γ 的每个链接 ℓ , 引入一个包含 ℓ "源节点" ℓ_- 的定向 2 曲面 S_ℓ , 令 ℓ 位于 S_ℓ "上方", ℓ_- 处所有其他链接位于 S_ℓ "下方", 再定义流量 $J_\ell^{IJ} := \int_{S_\ell} J^{IJ} \in \mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$ 。那么由 (30) 可得泊松括号 [61]

$$\{J_\ell^{IJ}, h_{\ell'}\} = \delta_{\ell\ell'} h_\ell \tau^{IJ} \tag{33}$$

where $\tau^{IJ} = -\tau^{JI} \in \mathfrak{sl}(2,\mathbb{C})$ is given by $\tau^{0i} := \frac{1}{2}\sigma_i$ and $\tau^{ij} := \frac{i}{2}\varepsilon^{ikj}\sigma_k$ with σ_i the usual Pauli matrices. All components of each h_ℓ trivially Poisson commute with each other, which, together with (33) and the Jacobi identity, yields the further non-trivial Poisson bracket [62]

其中 $\tau^{IJ} = -\tau^{JI} \in \mathfrak{sl}(2,\mathbb{C})$ 由 $\tau^{0i} := \frac{1}{2}\sigma_i$ 和 $\tau^{ij} := \frac{i}{2}\varepsilon^{ikj}\sigma_k$ 给出，σ_i 是常用泡利矩阵。每个 h_ℓ 的所有分量之间均平凡泊松对易，结合式 (33) 和雅可比恒等式，可得到如下额外的非平凡泊松括号 [62]

$$\{J_\ell^{IJ}, J_{\ell'}^{KL}\} = \delta_{\ell\ell'} C^{IJKL}{}_{MN} J_\ell^{MN}, \tag{34}$$

where $C^{IJKL}{}_{MN}$ are the structure constants of $\mathfrak{so}(3,1)$. All other Poisson brackets are zero. These Poisson brackets can also be derived from a discrete action [46,63]. They give the discrete boundary phase space $\Gamma := \left\{(h_\ell, J_\ell)_{\ell\in\gamma}\right\}$ the structure of the cotangent bundle over $\mathcal{C} \cong \mathrm{SL}(2,\mathbb{C})^L$,

其中 $C^{IJKL}{}_{MN}$ 是 $\mathfrak{so}(3,1)$ 的结构常数。其余所有泊松括号均为零。这些泊松括号也可由离散作用量推导得到 [46,63]。它们为离散边界相空间 $\Gamma := \left\{(h_\ell, J_\ell)_{\ell\in\gamma}\right\}$ 赋予了 $\mathcal{C} \cong \mathrm{SL}(2,\mathbb{C})^L$ 上余切丛的结构,

$$T^*\mathrm{SL}(2,\mathbb{C})^L := \left\{(h_\ell, \mu_\ell)_{\ell\in\gamma} \mid \mu_\ell \in T^*_{h_\ell}\mathrm{SL}(2,\mathbb{C})\right\} \tag{35}$$

in which J_ℓ^{IJ} is identified with $\langle \mu_\ell, L_{\tau^{IJ}} \rangle$, where L_x denotes the left-invariant vector field associated with the algebra element x . Quantization yields the boundary Hilbert space

其中 J_ℓ^{IJ} 等同于 $\langle \mu_\ell, L_{\tau^{IJ}} \rangle$ ，L_x 表示对应于代数元 x 的左不变向量场。量子化后得到边界希尔伯特空间

$$\mathcal{H}_{\partial F} = L_2\left(\mathrm{SL}(2,\mathbb{C})^L\right) \tag{36}$$

with basic operators $\left(\hat{h}_\ell\right)^A{}_B$ acting by multiplication and $\hat{J}_\ell^{IJ}$ as the derivative operator corresponding to $L_{\tau^{IJ}}$ on the copy of the group corresponding to ℓ . The commutators of the basic operators mimic exactly the Poisson brackets of their classical counterparts. In particular, this means that, as in loop quantum gravity, the fluxes do not commute, but rather satisfy the same algebra as the gauge group, sl(2, $\mathbb{C}$).

其中基本算符 $\left(\hat{h}_\ell\right)^A{}_B$ 通过乘法作用，$\hat{J}_\ell^{IJ}$ 是对应于 $L_{\tau^{IJ}}$ 的导数算符，作用在对应于 ℓ 的群副本上。基本算符的对易关系完全复刻了经典对应量的泊松括号。这尤其说明，和圈量子引力一样，流量不对易，而是满足和规范群 sl(2, $\mathbb{C}$) 完全相同的代数。

If we interpret the equation of motion (29) to be a priori satisfied along the interior of each link ℓ (but not at its endpoints), then, in addition to the flux J_ℓ at the source node of ℓ , we can define the flux at the target node of ℓ as the parallel transport of J_ℓ via h_ℓ . That is, for n the source of ℓ and n' the target of ℓ , define

如果我们认为运动方程 (29) 先验地在每条链接 ℓ 的内部满足 (但在端点不满足)，那么除了 ℓ 源节点处的流量 J_ℓ ，我们还可以通过 h_ℓ 平行移动将流量定义在 ℓ 的目标节点。也就是说，设 n 是 ℓ 的源，n' 是 ℓ 的靶，定义

$$J_\ell(n) := J_\ell,\ J_\ell(n') := h_\ell \triangleright J_\ell \tag{37}$$

where $\triangleright$ denotes the adjoint action. In terms of the cotangent bundle structure of the phase space (35), $J_\ell(n')$ then takes the form $\langle \mu_\ell, R_{\tau IJ} \rangle$ with R_x the right-invariant vector field corresponding to x . Quantization of the expression for $J_\ell(n')$ in (37) then leads to $\hat{J}^{IJ}_\ell(n')$ acting as the derivative operator corresponding to $R_{\tau IJ}$ on the copy of $\mathrm{SL}(2,\mathbb{C})$ corresponding to h_ℓ .

其中 $\triangleright$ 表示伴随作用。就相空间 (35) 的余切丛结构而言，$J_\ell(n')$ 可写为 $\langle \mu_\ell, R_{\tau IJ} \rangle$ ，其中 R_x 是对应于 x 的右不变向量场。对 (37) 中 $J_\ell(n')$ 的表达式量子化后，得到 $\hat{J}^{IJ}_\ell(n')$ 作为导数算符作用在对应于 h_ℓ 的 $\mathrm{SL}(2,\mathbb{C})$ 副本上，对应于 $R_{\tau IJ}$ 。

For each link ℓ and node n adjacent, the operators $\hat{J}^{IJ}_\ell(n)$ satisfy the Lorentz algebra, with rotation and boost generators given by

对每条链接 ℓ 和与之相邻的节点 n ，算符 $\hat{J}^{IJ}_\ell(n)$ 满足洛伦兹代数，其旋转生成元和 boost 生成元由下式给出

$$\hat{L}^i_\ell(n) := -\frac{1}{2}\varepsilon^i{}_{jk}\hat{J}^{jk}_\ell(n),\ \hat{K}^i_\ell(n) := \hat{J}^{0i}_\ell(n). \tag{38}$$

For a given link ℓ , there are two sets of the above generators, one for each node n, n' , acting as the left- and right-invariant vector fields on the corresponding group element h_ℓ . The jm and ln subscripts of the $\mathrm{SL}(2,\mathbb{C})$ D-matrices of the last subsection, evaluated for $h = h_\ell$, correspond to these two sets of generators. From the previous section, the corresponding Casimir operators are

对于给定的链接 ℓ ，存在上述生成元的两个集合，分别对应每个节点 n, n' ，在对应群元 h_ℓ 上作用为左不变和右不变向量场。上一小节中针对 $h = h_\ell$ 求值的 $\mathrm{SL}(2,\mathbb{C})$ D 矩阵的 jm 和 ln 下标就对应这两组生成元。由上一节可知，对应的卡西米尔算符为

$$\hat{C}_{1,\ell} := \hat{L}_\ell(n)^2 - \hat{K}_\ell(n)^2,\ \hat{C}_{2,\ell} := \hat{L}_\ell(n) \cdot \hat{K}_\ell(n). \tag{39}$$

Since the fluxes on either side of a given link ℓ are related by parallel transport, they yield the same Casimirs, so that the above operators are independent of n . The simultaneous eigenvalues of the above operators, as well as $\hat{L}_\ell(n)^2$ and $\hat{L}_\ell(n)_z$, are then parameterized by the set of quantum numbers $(\rho_\ell, k_\ell, j_{\ell n}, m_{\ell n})$. For each such set of quantum numbers for each ℓ and $n \in \ell$, there is a state $\psi_{\{\rho_\ell, k_\ell, j_{\ell n}, m_{\ell n}\}}$ in the boundary Hilbert space, unique up to phase, satisfying the equations

由于给定链接 ℓ 两侧的流通过平行移动物理关联，它们给出相同的卡西米尔，因此上述算符与 n 无关。上述算符以及 $\hat{L}_\ell(n)^2$ 和 $\hat{L}_\ell(n)_z$ 的共同本征值可由量子数集合 $(\rho_\ell, k_\ell, j_{\ell n}, m_{\ell n})$ 参数化。对于每个 ℓ 和 $n \in \ell$ 的每一组这样的量子数，边界希尔伯特空间中都存在一个满足以下方程的态 $\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}}$，它在相位差范围内是唯一的

$$\widehat{C}_{1,\ell}\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}} = \left(k_\ell^2 - \rho_\ell^2 - 1\right)\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}}$$

$$\widehat{C}_{2,\ell}\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}} = \rho_\ell k_\ell \psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}}$$

$$\hat{L}_\ell(n)^2\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}} = j_{\ell n}\left(j_{\ell n}+1\right)\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}}$$

$$\hat{L}_\ell(n)_z\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}} = m_{\ell n}\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}} \tag{40}$$

given explicitly by

显式表达式为

$$\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}}\left(\{h_\ell\}\right) = \prod_\ell D^{(\rho_\ell,k_\ell)}_{j_{\ell\ell_+}m_{\ell\ell_+}j_{\ell\ell_-}m_{\ell\ell_-}}\left(h_\ell\right) \tag{41}$$

where ℓ_- and ℓ_+ denote the source and target of ℓ . These states are referred to as projected spin networks [64] and form an orthonormal basis of the boundary Hilbert space, as follows from a version of the Peter-Weyl theorem as applied to $\mathrm{SL}(2,\mathbb{C})^L$.

其中 ℓ_- 和 ℓ_+ 分别表示 ℓ 的源节点和目标节点。这些态被称为投影自旋网 [64]，构成边界希尔伯特空间的一组标准正交基，这可由应用于 $\mathrm{SL}(2,\mathbb{C})^L$ 的彼得-外尔定理的相关版本导出

The transition amplitude for such a boundary state is then given by

这类边界态的跃迁振幅可表示为

$$W_{\mathcal{K}}\left(\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}\right) = \int\left(\prod_{\ell\in\gamma}dh_\ell\right)\psi_{\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}}\left(\{h_\ell\}\right)W\left(\{h_\ell\}\right). \tag{42}$$

By substituting in (32), using the Plancherel theorem, inserting resolutions of the identity on each irrep (ρ,k) of $\mathrm{SL}\left(2,\mathbb{C}\right)$ in terms of the states $|\rho,k;j,m\rangle$, and dropping one overall factor per vertex equaling the (divergent) volume of $\mathrm{SL}\left(2,\mathbb{C}\right)$, one can show [66]

将 (32) 代入，利用普朗歇尔定理，对 $\mathrm{SL}\left(2,\mathbb{C}\right)$ 的每个不可约表示 (ρ,k) 插入以态 $|\rho,k;j,m\rangle$ 表示的单位分解，并且去掉每个顶点处一个等于 $\mathrm{SL}\left(2,\mathbb{C}\right)$ (发散) 体积的整体因子后，可以证明 [66]

$$W\left(\{\rho_\ell,k_\ell,j_{\ell n},m_{\ell n}\}\right) = \sum_{\{\rho_f,k_f,j_{fe},m_{fe}\}}\left(\prod_{f\in\mathcal{K}}A_f\right)\left(\prod_{v\in\mathcal{K}}A_v\right) \tag{43}$$

where A_f is a function only of $\{\rho_f, k_f, j_{fe}, m_{fe}\}_{e\in f}$ and A_v is a function only of $\{\rho_f, k_f, j_{fe}, m_{fe}\}_{e,f\ni v}$, given explicitly by

其中 A_f 是仅关于 $\{\rho_f, k_f, j_{fe}, m_{fe}\}_{e\in f}$ 的函数，A_v 是仅关于 $\{\rho_f, k_f, j_{fe}, m_{fe}\}_{e,f\ni v}$ 的函数，显式表达式为

$$
\begin{aligned}
A_v &= W_v\left(\{\rho_f, k_f, j_{fe}, m_{fe}\}_{e,f\ni v}\right) \\
&= \int \left(\prod_{e\ni v} dh_e\right) \prod_{f\ni v} D^{(\rho_f,k_f)}_{j_{fe} m_{fe}\, j_{fe'} m_{fe'}}\left(h_e^{-1} h_{e'}\right)
\end{aligned}
\tag{44}
$$

where e, e' are the two edges sharing the same face f at v and where $\tilde{e}$ is an arbitrary choice of edge at v . Note that A_v is just the BF transition amplitude (42), for the projective spin network associated with the labels $\{\rho_f, k_f, j_{fe}, m_{fe}\}_{v\in e\in f}$, in the boundary Hilbert space of the single vertex v . The quantum numbers $\{\rho_f, k_f, j_{fe}, m_{fe}\}$ thus again have the interpretation of being quantum numbers of flux operators $\hat{J}_f(e)$, this time associated with each face f in $\mathcal{K}$ and edge e therein. While the flux operators on the boundary are parallel transported along each link, these flux operators in the bulk are parallel transported around each face, so that the quantum numbers ρ_f, k_f depend only on f and not e [46]. Equation (44) is the vertex amplitude of the $\mathrm{SL}(2,\mathbb{C})$ BF theory. By imposing simplicity on the boundary Hilbert space of each vertex, one obtains a proposal for the vertex amplitude of loop quantum gravity.

其中 e, e' 是在 v 处共享同一个面 f 的两条边，$\tilde{e}$ 是在 v 处任意选取的一条边。注意，A_v 就是与标记 $\{\rho_f, k_f, j_{fe}, m_{fe}\}_{v\in e\in f}$ 关联的投影自旋网络在单顶点 v 的边界希尔伯特空间中的 BF 跃迁振幅 (42)。因此量子数 $\{\rho_f, k_f, j_{fe}, m_{fe}\}$ 再次被解释为通量算符 $\hat{J}_f(e)$ 的量子数，这一次它们对应 $\mathcal{K}$ 中的每个面 f 以及面 f 内的边 e 。边界上的通量算符沿每条链接平行移动，而体中的这些通量算符绕每个面平行移动，因此量子数 ρ_f, k_f 仅依赖于 f ，不依赖于 e [46]。式 (44) 是 $\mathrm{SL}(2,\mathbb{C})$ BF 理论的顶点振幅。通过对每个顶点的边界希尔伯特空间施加简单性条件，我们得到了圈量子引力顶点振幅的一个方案。

Quantum Simplicity

量子简单性

Consider now a discretization of the linear primary simplicity constraints (20), where each generator is associated with a pair node-link of each vertex graph:

现在我们来考虑线性主简单约束 (20) 的离散化，其中每个生成元对应顶点图的一对节点-链接:

$$
\mathbf{S} := \mathbf{K} - \gamma \mathbf{L} \approx 0. \tag{45}
$$

As before, we assume that all tetrahedra are space-like. Due to the non-commutativity of the fluxes (34), part of these simplicity constraints become second class even without taking the secondary ones into account. Dirac's original prescription for quantizing systems with constraints allows for only first-class constraints to be implemented as operator equations in the quantum theory, as implementing second-class constraints in this way would kill too many degrees of freedom and impose unphysical constraints. However, a unified method for imposing both first- and second-class constraints in quantum theory was later introduced by Thiemann - the master constraint method [67, 68] - in which one combines the constraints into a choice of positive definite quadratic form, yielding a single constraint that vanishes if and only if all the original constraints vanish. In our case, SU (2) invariance restricts the quadratic form to be simply the square of $\mathbf{S}$,

和之前一样，我们假设所有四面体都是类空的。由于流量 (34) 不对易，即使不考虑次级约束，这些简单约束中也有一部分成为第二类约束。狄拉克最初对带约束系统量子化的规则要求，只有第一类约束能作为算符方程引入量子理论，因为若按这种方式引入第二类约束，会消除过多自由度并施加非物理约束。但后来 Timann 提出了一种能在量子理论中同时施加第一类和第二类约束的统一方法，即主约束法 [67, 68] : 该方法将所有约束组合为一个正定二次型，得到的单一约束当且仅当所有原约束都为零时才为零。在我们的问题中，SU (2) 不变性要求二次型就是 $\mathbf{S}$ 的平方，

$$M \ := \ \mathbf{S}^2 = \left(1+\gamma^2\right)\mathbf{L}^2 - C_1 - 2\gamma C_2, \tag{46}$$

so that M is a linear combination of the two SL $(2,\mathbb{C})$ Casimirs and the one Casimir $\mathbf{L}^2$ for the subgroup, SU (2) . In quantizing the master constraint, one must allow for $\hbar$-scale modifications to ensure a sufficiently large space of solutions. This leads (46) to be quantized as

因此 M 是两个 SL $(2,\mathbb{C})$ 卡西米爾和一个子群卡西米爾 $\mathbf{L}^2$ 的线性组合，即 SU (2) 。对主约束量子化时，必须允许 $\hbar$ 尺度的修正，以保证足够大的解空间。这使得 (46) 被量子化为

$$\widehat{M}\,|\rho,k;j,m\rangle \ := \ \left(\left(1+\gamma^2\right)k^2 - \left(k^2-\rho^2\right) - 2\rho k\right)|\rho,k;j,m\rangle$$

$$= \left(\left(1+\gamma^2\right)\left(j^2-k^2\right) + \left(\rho-\gamma k\right)^2\right)|\rho,k;j,m\rangle. \tag{47}$$

Since both terms are non-negative definite, setting this equal to zero implies both

由于两项都是非负定的，令其等于零即可得到两个条件

$$k = j \ \text{ and } \ \rho = \gamma k. \tag{48}$$

We refer to these representations as γ -simple. A different convention with $-\gamma$ also appears in the literature, the relation being simply complex conjugation of all representations. The map from the spin j representation of SU(2) to the corresponding γ -simple representation $(\rho,k) = (\gamma j, j)$ of SL $(2,\mathbb{C})$ is called the Y -map in the early EPRL literature. (The difference with the BC model is that the latter imposes simplicity of J , as opposed to γ -simplicity. As a result, the only allowed irreps are either $\rho = 0$ or $k = 0$ and are interpreted as describing a model with space-like or time-like faces, respectively.)

我们将这些表示称为 γ -简单表示。文献中也存在使用 $-\gamma$ 的不同约定，二者的关系仅仅是所有表示的复共轭。从 SU(2) 的自旋 j 表示对应到 SL $(2,\mathbb{C})$ 的 γ -简单表示 $(\rho,k) = (\gamma j, j)$ 的映射，在早期 EPRL 文献中被称为 Y 映射。(和 BC 模型的区别在于:BC 模型要求的是 J 简单性，而非 γ 简单性，因此 BC 模型只允许不可约表示 $\rho = 0$ 或 $k = 0$ ，分别被解释为描述类空面模型和类时面模型。)

The resulting matrix elements satisfy a sort of analytic property: they are fully determined by their restriction to $\mathrm{SU}(2)$, via an integral kernel defined in terms of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SU}(2)$ characters [69, 70] ,

最终得到的矩阵元满足一类解析性质: 通过一个由 $\mathrm{SL}(2,\mathbb{C})$ 和 $\mathrm{SU}(2)$ 特征标 [69, 70] 定义的积分核，矩阵元完全由其在 $\mathrm{SU}(2)$ 上的限制确定,

$$D^{(\rho,j)}_{jmjn}(h) = \int_{\mathrm{SU}(2)} dg K(g,h) D^{(j)}_{mn}(g), \; K = \sum_j d_j^2 \int_{\mathrm{SU}(2)} dk \chi^{(\rho,j)}(gk) \chi^{(j)}(kh).$$

This property can also be understood in classical terms, recalling that these representations are a quantization of the canonical manifold $T^*\mathrm{SL}(2,\mathbb{C})$. First, one can split the three equations (45) into a first-class Lorentz-invariant constraint, given by $\gamma\left(\mathbf{L}^2-\mathbf{K}^2\right)=\left(1-\gamma^2\right)\mathbf{K}\cdot\mathbf{L}$, and two second-class constraints (a nice explicit decomposition can be obtained working with spinors [71]). The first involves the Casimirs, and it is thus the same among left- and right-invariant generators, whereas the second has to be implemented independently on both. This gives a total of one first-class constraint and four second-class constraints. It can be shown that symplectic reduction gives a 6d canonical submanifold isomorphic to $T^*\mathrm{SU}(2)$ [71], with holonomy g . The analytic property mentioned above is the quantum manifestation of this classical reduction.

这个性质也可以从经典层面理解: 我们知道这些表示是正则流形 $T^*\mathrm{SL}(2,\mathbb{C})$ 的量子化。首先，可以将三个方程 (45) 拆分为一个洛伦兹不变的第一类约束 $\gamma\left(\mathbf{L}^2-\mathbf{K}^2\right)=\left(1-\gamma^2\right)\mathbf{K}\cdot\mathbf{L}$ ，和两个第二类约束 (使用旋量处理可以得到简洁的显式分解 [71])。第一类约束涉及卡西米爾，因此左右不变生成元的约束形式相同，而第二类约束需要在两边分别独立引入，总共得到一个第一类约束和四个第二类约束。可以证明，辛约化给出一个同构于 $T^*\mathrm{SU}(2)$ 的 6 维正则子流形 [71]，带有和乐 g 。上文提到的解析性质就是这种经典约化在量子层面的体现。

There is another useful aspect that can be learned from this reduction. Imposing the constraints but not dividing by the orbits, one has a 7d space whose extra dimension is spanned by a real variable Ξ that spans each orbit of the first-class constraint. One can then construct a general $\mathrm{SL}(2,\mathbb{C})$ holonomy $h(g,\Xi)$ in this space. In the absence of secondary constraints, we can treat Ξ as gauge and remove it, for instance, gauge fixing it to 0 so to reduce trivially h to g . If secondary constraints are present, their solution can always be seen as providing a specific gauge fixing in terms of the reduced data - namely, a non-trivial section $\Xi(g,X)$, the discrete analogue of the mapping $K=K(A,E)$ they provide in the continuum. One can then argue that g provides the correct discrete analogue of the Ashtekar-Barbero (AB) connection [71], whence the γ -simple matrices provide a quantum version of the map between $\mathrm{SL}(2,\mathbb{C})$ holonomies and AB holonomies, provided the correct non- trivial section is used. These considerations can be useful to distinguish the role of the two different connections present in the model, the spin connection and the AB connection.

通过这一约化我们还能得到另一个有用结论。施加约束但不对轨道做商化处理后，我们会得到一个 7 维空间，其额外维度由实变量 Ξ 张成，该变量张成第一类约束的每条轨道。随后我们可以在这个空间中构造一般的 $\mathrm{SL}(2,\mathbb{C})$ 和乐 $h(g,\Xi)$ 。在没有次级约束的情况下，我们可以将 Ξ 视为规范并将其移除，例如将其规范固定为 0，从而将 h 平凡约化为 g 。如果存在次级约束，它们的解总能被解读为针对约化数据给出了一个特定规范固定——即一个非平凡截面 $\Xi(g,X)$ ，这是它们在连续统中提供的映射 $K=K(A,E)$ 的离散类比。由此可以说明，当采用正确的非平凡截面时，g 是阿西卡-巴贝罗 (AB) 联络 [71] 的正确离散类比，因此 γ -单矩阵给出了 $\mathrm{SL}(2,\mathbb{C})$ 和乐与 AB 和乐之间映射的量子版本。这些结论有助于区分模型中两种不同联络——自旋联络与 AB 联络各自的作用。

Definition of the 4-Simplex EPRL Amplitude

4-单形 EPRL 振幅的定义

At this point, we return to the quantum BF theory introduced in section ”Classical and Quantum BF Theory in Four Dimensions”, but now require that the cell complex $\mathcal{K}$ be dual to a simplicial complex. In particular, each vertex is now dual to a 4-simplex, and one can consider the BF amplitude (44) for a given set of data on the boundary of such a 4-simplex. By restricting this boundary data to satisfy the quantum simplicity constraints (48), the boundary data reduce to those for a single vertex in loop quantum gravity (13) and yield the EPRL vertex amplitude:

至此，我们回到“四维经典与量子 BF 理论”小节介绍的量子 BF 理论，但现在要求胞复形 $\mathcal{K}$ 对偶于一个单纯复形。具体而言，每个顶点现在对偶于一个 4-单形，我们可以对这类 4-单形边界上的一组给定数据考虑 BF 振幅 (44)。通过限制该边界数据满足量子简单性约束 (48)，边界数据可约化为圈量子引力中单个顶点的数据 (13)，并得到 EPRL 顶点振幅:

$$A_v\left(j_f, m_{fe}\right) = \int \left(\prod_e dh_e\right) \delta\left(h_{\tilde{e}}\right) \prod_f D^{(\gamma j_f, j_f)}_{j_f m_{fe} j_f m_{e'f}}\left(h_e^{-1} h_{e'}\right), \tag{49}$$

where $\tilde{e}$ is an arbitrary choice of edge at v . One of the five group integrals over the non-compact manifold $\mathrm{SL}(2, \mathbb{C})$ is redundant and has to be dropped to ensure finiteness of the amplitude [72]. This is achieved by the Dirac delta, which is with respect to the Haar measure $dh_{\tilde{e}}$ and peaked at the identity. The definition is valid only for $j_f \neq 0$. If a face has zero spin, the matrix should be replaced by 1 . (This definition is necessary because $D^{(0,0)}(h) \neq 1$.) The model is then completed taking trivial edge amplitudes and $2j_f + 1$ for the face amplitudes in (13). We first review results concerning a single vertex and later on comment on multi-vertex amplitudes. Multi-vertex analysis is crucial for understanding curvature, dynamics, and questions related to infrared divergences and causality; however, most results of this analysis are not yet fully settled, so that we will mostly limit ourselves to point to the relevant research directions and literature.

其中 $\tilde{e}$ 是在 v 处任意选取的一条边。五个非紧流形 $\mathrm{SL}(2, \mathbb{C})$ 上的群积分中有一个是冗余的，必须舍去以保证振幅有限 [72]，这由狄拉克 δ 函数实现，该 δ 函数关于哈尔测度 $dh_{\tilde{e}}$ 定义，并在单位元处取峰值。该定义仅对 $j_f \neq 0$ 成立。若一个面的自旋为零，对应矩阵应替换为 1。(该定义是必要的，因为 $D^{(0,0)}(h) \neq 1$ 。) 之后模型补全，在 (13) 中采用平凡边振幅与 $2j_f + 1$ 作为面振幅。我们先回顾单个顶点的相关结果，之后讨论多顶点振幅。多顶点分析对理解曲率、动力学以及红外发散和因果性相关问题至关重要；但该分析的大多数结果尚未完全定论，因此我们大多仅指明相关研究方向与文献。

When working with a single 4-simplex, it is often convenient to simplify the notation and use $a = 1, \dots 5$ to label the edges, namely, the nodes of the vertex graph Γ , and (ab) for the faces, namely, the 10 links between the nodes (we will no longer need these Latin letters for hypersurface tensors). $\tilde{e}$ will then be chosen to correspond to $a = 1$, so that the integrals (49) are only over h_a with $a = 2, \dots 5$. In the following, we will alternate between the two notations, with the dictionary just spelled out in mind.

研究单个 4-单形时，简化记号通常更为方便: 用 $a = 1, \ldots 5$ 标记边，即顶点图 Γ 的节点，用 (ab) 标记面，即节点之间的 10 条连线 (我们不再用这些拉丁字母表示超曲面张量)。之后 $\tilde{e}$ 会被选为对应 $a = 1$，因此积分 (49) 仅对 h_a 在 $a = 2, \ldots 5$ 条件下积分。在下文中我们会交替使用两种记号，请记住刚刚给出的对应规则。

Notice that the Y -map is imposed only at the tetrahedra: If we split the product $h_a^{-1} h_b$ using the group product, we obtain arbitrary weights:

请注意，Y 映射仅施加在四面体上: 如果我们利用群乘积拆分乘积 $h_a^{-1} h_b$，会得到任意权重:

$$D_{jmjn}^{(\gamma j, j)}\left(h_a^{-1} h_b\right) = \sum_{l=j}^{\infty} \sum_{p=-l}^{l} D_{jmlp}^{(\gamma j, j)}\left(h_a^{-1}\right) D_{lpjn}^{(\gamma j, j)}\left(h_b\right). \tag{50}$$

If one imposes additional Y -maps on the internal weights, one obtains a toy model, closely related to SU(2) BF theory and dubbed simplified EPRL model in [73], which is useful for numerical investigations with Euclidean boundary data.

如果在内部权重上额外施加 Y 映射，就会得到一个玩具模型，它与 SU(2) BF 理论密切相关，在文献 [73] 中被称为简化 EPRL 模型，对欧几里得边界数据的数值研究很有用。

Properties

性质

Covariance The integral (49) defines (the lowest weights of) an $\mathrm{SL}(2, \mathbb{C})$ invariant tensor and can be expressed in terms of the corresponding Clebsch-Gordan coefficients. Thanks to the Y -map, the quantum labels are in one-to-one correspondence with SU(2) weights; hence, it can be used to provide transition amplitudes for LQG with 4-valent spin networks. Bulk Lorentz invariance and boundary covariance were proven in [70].

协变性积分 (49) 定义了一个 $\mathrm{SL}(2, \mathbb{C})$ 不变张量 (其最低权)，可通过相应的克莱布希-高登系数表示。借助 Y 映射, 量子标签与 SU(2) 权一一对应; 因此该模型可用于为 4 价自旋网络的圈量子引力 (LQG) 提供跃迁振幅。整体洛伦兹不变性与边界协变性已在文献 [70] 中得到证明。

Furthermore, the vertex amplitude is manifestly invariant under global SU(2) transformations at each edge. Therefore, it can be contracted with invariant inter-twiner tensors without any loss of information, giving the spin-intertwiner version

此外，顶角振幅在每条边的整体 SU(2) 变换下明显不变。因此它可以和不变纠缠张量缩并而不损失任何信息，得到自旋-纠缠版本

$$A_v\left(j_f, i_e\right) = \sum_{m_{fe}} I_{j_f m_{fe}}^{(i_e)} A_v\left(j_f, m_{fe}\right). \tag{51}$$

The intertwiner label i_e depends on a choice of recoupling basis, and I is an orthonormal basis for intertwining tensors such as Wigner's 3jm symbols.

纠缠子标签 i_e 依赖于重耦合基的选择，而 I 是纠缠张量的标准正交基，例如维格纳 3jm 符号。

Coherent states The boundary magnetic indices can be superimposed to obtain SU(2) coherent states, as suggested initially in [65]. This allows a simple geometric interpretation of the amplitudes and gives a useful handle to study the dynamics. To define the coherent amplitude, we take lowest weight coherent states, labeled by $\mathbf{n}_{ab}$ at the source and $-\mathbf{n}_{ba}$ at the target of each link. A standard calculation exploiting the factorization property of the coherent states leads to [74-76]

相干态按照文献 [65] 最初的提议，可以叠加边界磁指标得到 SU(2) 相干态。这为振幅提供了简单的几何诠释，也为研究动力学提供了有用的工具。为定义相干振幅，我们取最低权相干态，每条边的源点和目标点分别由 $\mathbf{n}_{ab}$ 和 $-\mathbf{n}_{ba}$ 标记。利用相干态因式分解性质的标准计算可得 [74-76]

$$A_v\left(j_{ab}, \mathbf{n}_{ab}, -\mathbf{n}_{ba}\right) = e^{i \sum\limits_{(ab)} j_{ab}\psi_{ab}} \int \prod_{a=2}^{N} dh_a \int \prod_{(ab)} \frac{d\mu\left(z_{ab}\right)}{\left\|h_a^{\dagger} z_{ab}\right\|^2 \left\|h_b^{\dagger} z_{ab}\right\|^2} \exp S,$$

(52)

where the action is

其中作用量为

$$S(h, z, \zeta) := \sum_{(ab)} j_{ab} \ln \frac{\left\langle \zeta_{ab} \mid h_a^{\dagger} z_{ab} \right\rangle^2 \left\langle h_b^{\dagger} z_{ab} \mid \zeta_{ba} \right]^2}{\left\|h_a^{\dagger} z_{ab}\right\|^2 \left\|h_b^{\dagger} z_{ab}\right\|^2} + i\gamma j_{ab} \ln \frac{\left\|h_b^{\dagger} z_{ab}\right\|^2}{\left\|h_a^{\dagger} z_{ab}\right\|^2}, \tag{53}$$

with $h_1 = \mathbb{1}$. In this expression, there are two types of spinors, Latin and Greek. The Latin spinors $|z_{ab}\rangle$ provide the homogeneous realization of the infinite-dimensional $\mathrm{SL}(2,\mathbb{C})$ irreps. The integrand is invariant under complex rescalings $|z_{ab}\rangle \mapsto \lambda_{ab}|z_{ab}\rangle$, and the spinorial integration is defined over $\mathbb{C}P^1 \cong S^2$. See [75,77] for more details. The Greek spinors are determined by the boundary normals via the map $(\mathbf{n}_{ab}, -\mathbf{n}_{ba}) \mapsto (|\zeta_{ab}\rangle, |\zeta_{ba}])$. This map is not unique, and its choice determines the overall phase ψ ; see Appendix of [76] for details.

其中 $h_1 = \mathbb{1}$。该表达式包含两类旋量: 拉丁标号旋量和希腊标号旋量。拉丁标号旋量 $|z_{ab}\rangle$ 给出了无限维 $\mathrm{SL}(2,\mathbb{C})$ 不可约表示的齐次实现。被积函数在复缩放 $|z_{ab}\rangle \mapsto \lambda_{ab}|z_{ab}\rangle$ 下不变, 旋量积分定义在 $\mathbb{C}P^1 \cong S^2$ 上,更多细节见文献 [75,77]。希腊标号旋量由边界法向通过映射 $(\mathbf{n}_{ab}, -\mathbf{n}_{ba}) \mapsto (|\zeta_{ab}\rangle, |\zeta_{ba}])$ 确定。该映射不唯一，其选择决定了整体相位 ψ，细节见文献 [76] 的附录。

An important property of the coherent amplitude is that its norm is invariant under rotations of the boundary data at each node. In other words, its norm depends only on rotational-invariant quantities such as the angles between the normals, whereas its phase depends also on the orientation of the normals.

相干振幅的一个重要性质是，其范数在每个节点的边界数据转动下保持不变。换句话说，范数仅依赖于法向夹角这类转动不变量，而相位还依赖于法向的朝向。

Representation in twistor space The amplitude can be derived as an integral in twistor space [71, 78] . This is obtained through the parametrization of each copy of $T^*\mathrm{SL}(2,\mathbb{C})$ in the phase space (35) in terms of a pair of twistors with matching SL$(2,\mathbb{C})$ Casimirs, and it is useful to provide both a bridge to twistor theory and a complementary derivation of the vertex amplitude from a discrete BF action. The twistorial description offered an independent insight into the flatness issue [79], mathematical tools to describe time-like [80] and null [81] hypersurfaces in spin foams, and potentially different implementations of the simplicity constraints along the lines suggested in [82,83]. It also spurred a brief attempt at studying more the relation between the tools used in LQG and twistors [84].

扭量空间表示振幅可以推导为扭量空间 [71, 78] 上的积分。这一结果是通过对相空间 (35) 中每个 $T^*\mathrm{SL}(2,\mathbb{C})$ 拷贝，用一对满足 SL$(2,\mathbb{C})$ 卡西米尔不变量匹配条件的扭量参数化得到的，它一方面搭建了通往扭量理论的桥梁，另一方面也补充了从离散 BF 作用量推导顶角振幅的过程。扭量描述为平坦性问题提供了独立视角 [79]，也给出了描述自旋泡沫中类时 [80] 和类光 [81] 超曲面的数学工具，甚至可以按照文献 [82,83] 的思路，以不同方式实现简单性约束。它也推动了少量研究，进一步探索圈量子引力所用工具与扭量之间的关系 [84]。

Large Spin Asymptotics

大自旋渐近

The action (53) has two useful properties: its real part has upper bound 0, and it is linear in the spins. The latter means that the integral can be approximated using saddle point techniques in the limit of large spins. We consider only the case in which all spins are homogeneously large. Inhomogeneous limits have been studied for SU(2) amplitudes; see, e.g., [85, 86]. The result of this analysis shows that the vertex amplitude decays exponentially, unless the boundary data satisfy some special configurations, and then the decay is power law. The power law decay requires boundary data such that the action admits a dominant saddle point, namely, a point such that the gradient of the action with respect to all integration variables vanishes and such that the real value of the action at the saddle point takes its maximal value. To understand the meaning of the special configurations, it is useful to interpret the boundary data in terms of discrete geometries. This can be done as follows. On each oriented link, we interpret the triple $(j_{ab}, \mathbf{n}_{ab}, \mathbf{n}_{ba})$ as the area and the normal directions of the triangle dual to the link in the frame of the tetrahedron dual to the source and target of the link. These variables parametrize the subset of the holonomy-flux phase space $T^*\mathrm{SU}(2)$ with vanishing twist angle ξ_{ab} [87]. (The reason one considers boundary data with vanishing twist angles is that the amplitudes are coherent in the directions, but sharp in the areas. One can consider superpositions of vertex amplitudes also in the spins, like it is done in the propagator calculations [88]. These will be labeled also by the twist angles, and the boundary data then parametrize the complete holonomies and fluxes.) A special feature of a flat 4-simplex is that the Jacobian between edge lengths and areas is generically invertible. Therefore, one can use the spins alone to determine a unique Euclidean 4-simplex (the spins satisfy the Euclidean triangle inequalities), a unique Lorentzian one (the spins satisfy the Lorentzian triangle inequalities), or no 4-simplex at all (no triangle inequalities are fully satisfied). Therefore, whether the power law behavior exists depends entirely on whether the unit vectors are suitably adapted to the spins. To that end, we list some special subsets of the boundary data which are relevant to the saddle point analysis.

作用量 (53) 有两个有用性质: 其实部上界为 0，且对自旋是线性的。后者意味着可以在大自旋极限下用鞍点法近似积分。我们仅考虑所有自旋齐次增大的情况；非齐次极限已针对 SU(2) 振幅开展研究，参见例如文献 [85, 86]。该分析的结果表明，顶点振幅指数衰减，除非边界数据满足某些特殊构型，此时衰减为幂律。幂律衰减要求边界数据满足: 作用量存在主导鞍点，即该点处作用量对所有积分变量的梯度为零，且作用量在该鞍点的实部取最大值。为理解特殊构型的物理意义，将边界数据用离散几何解释是十分有用的，方式如下: 在每条有向链上，我们将三元组 $(j_{ab}, \mathbf{n}_{ab}, \mathbf{n}_{ba})$ 解释为源四面体和靶四面体坐标系中，对偶于该链的三角形的面积和法向方向。这些变量参数化了挠角 ξ_{ab} 为零的全纯-通量相空间 $T^*\mathrm{SU}(2)$ [87]。(人们考虑挠角为零的边界数据，原因在于振幅在方向上是相干的，但在面积上是尖锐的；也可以像传播子计算 [88] 中那样，考虑自旋空间内顶点振幅的叠加，这些叠加也会用挠角标记，此时边界数据可以参数化完整的全纯和通量。) 平坦 4 单形的一个特殊性质是，边长和面积之间的雅可比矩阵一般是可逆的。因此，仅用自旋就可以确定唯一的欧几里得 4 单形 (自旋满足欧几里得三角不等式)、唯一的洛伦兹 4 单形 (自旋满足洛伦兹三角不等式)，或是不存在任何 4 单形 (三角不等式未完全满足)。因此，幂律行为是否存在完全取决于单位向量是否适配于自旋。为此，我们列出边界数据中与鞍点分析相关的一些特殊子集。

(i) Twisted geometries. (Or closed twisted geometries, if the name twisted geometries is used for the generic parametrization of $T^*\mathrm{SU}(2)$ prior to imposing the closure conditions.) These are the data satisfying the closure conditions

(i) 扭曲几何。(若“扭曲几何”用于指代施加闭合条件前 $T^*\mathrm{SU}(2)$ 的一般参数化，则此处是闭合扭曲几何。) 这些是满足闭合条件的数据

$$\sum_{b\neq a} j_{ab}\mathbf{n}_{ab} = 0, \ \forall a. \tag{54}$$

Notice here the advantage of choosing the minus sign for the normals at the targets in (52). This allows us to write the closure constraints above without further need of specifying orientations. These data describe a tetrahedron with areas given by j_{ab} and 3 d dihedral angles

请注意，此处我们在 (52) 中为靶处的法向选择负号是有优势的，这让我们可以直接写出上述闭合约束，无需额外指定定向。这些数据描述了一个四面体，其面积由 j_{ab} 给出，二面角由 3 d 给出

$$\mathbf{n}_{ab} \cdot \mathbf{n}_{ac} = \cos\phi^a_{bc}. \tag{55}$$

Looking at a single node, this information allows one to reconstruct the entire geometry of a flat tetrahedron: four areas and two independent dihedral angles among the six possible ones. This gauge-invariant information defines the shape of the tetrahedron. The collection of such tetrahedra defines a twisted geometry. The geometry is twisted because the shapes of the triangles don't necessarily match when computed in the source or target frames. It has on the other hand a well-defined notion of signature, depending on whether the (edge-dependent) 4d dihedral angles θ^a_{bc} that can be constructed from the ϕ s using the spherical cosine laws satisfy Euclidean or Lorentzian inequalities.

对单个节点来说，该信息可以让我们重建平坦四面体的完整几何: 六个可能的二面角中的四个面积和两个独立二面角。该规范不变信息定义了四面体的形状。这类四面体的集合定义了一个扭曲几何。该几何是扭曲的，原因在于在源框架和靶框架中计算得到的三角形形状不一定匹配。另一方面，它有明确定义的符号差，取决于通过球面余弦定理从 ϕ 构造得到的 (依赖边的) 四维二面角 θ^a_{bc} 满足欧几里得不等式还是洛伦兹不等式。

For these data, there still is no dominant critical point; hence, the amplitude decays exponentially.

对于这类数据，仍然不存在主导临界点；因此振幅仍然指数衰减。

(ii) Vector geometries.

(ii) 向量几何。

These were introduced in [89] as the set satisfying closure and additionally the orientation conditions

这类几何在文献 [89] 中被提出，是满足闭合条件且额外满足定向条件的集合

$$R_a \mathbf{n}_{ab} = -R_b \mathbf{n}_{ba},\ R_a \in \mathrm{SO}(3)\,. \tag{56}$$

A gauge-invariant description of vector geometries is not known. A partial answer in terms of four shape parameters and one non-gauge-invariant angle is given in [90]. These conditions can be satisfied only for twisted geometries that have a Euclidean signature everywhere.

目前尚不清楚如何用规范不变的方式描述向量几何。文献 [90] 给出了用四个形状参数和一个非规范不变角得到的部分结果。这类条件仅对处处具有欧几里得符号差的扭曲几何成立。

If (and only if) the spins are Euclidean, these data admit a (unique) dominant saddle point. The amplitude then decays with the power law j^{-12} and a phase that depends on the orientation of the data.

当且仅当自旋是欧几里得的，这类数据才存在 (唯一的) 主导鞍点。此时振幅按幂律 j^{-12} 衰减，相位依赖于数据的定向。

(iii) Regge geometries.

(iii) 里奇几何。

These are data such that the shapes of the triangles match. In this case, there is a well-defined notion of edge lengths of the triangulation, and these define a 3d Regge geometry. Furthermore, since every triangulation of S^3 by 5 tetrahedra is flat-embeddable, the same data also define a unique 4-simplex, which can be either Euclidean or Lorentzian. The shape-matching also guarantees that both the 3d dihedral angles and the 4d ones defined from the spherical cosine laws are edge independent; hence, we can write the relation between them as $\theta_{ab}\ .\ (\phi)$.

这些数据满足三角形形状匹配条件。在这种情况下，三角剖分的边长有明确的定义，它们由此定义出一个三维里奇几何。此外，由于 S^3 的任何三角剖分都可由 5 个四面体平坦嵌入，这些相同的数据也能定义出唯一的四维单形，其既可以是欧几里得型也可以是洛伦兹型。形状匹配还保证了由球面余弦定律定义的三维二面角和四维二面角都与边无关；因此，我们可以将它们之间的关系写为 θ_{ab} 。(ϕ)。

For these data, there are two distinct dominant saddle points. The amplitude then decays with the power law j^{-12} , a global phase that depends on the orientation of the data, and a relative phase which is proportional to the Regge action

对于这些数据，存在两个不同的主导鞍点。振幅随后按幂律 j^{-12} 衰减，带有一个依赖于数据取向的整体相位，以及一个与里奇作用量成正比的相对相位

$$S\left(j_{ab},\phi_{bc}^{a}\right)=\gamma\sum_{ab}j_{ab}\theta_{ab}\left(\phi\right). \tag{57}$$

Numerical investigations show that the Hessians at the critical points are related by complex conjugation [76]; therefore, the relative phase can be written as a cosine.

数值研究表明，临界点处的海森矩阵通过复共轭关联 [76]；因此，相对相位可以写为余弦形式。

From the interpretation of the critical points, one can also conveniently group these types of data into ”degenerate” (vector and Euclidean Regge) and ”non-degenerate” (Lorentzian Regge) (see section ”Proper Vertex”).

根据对临界点的诠释，我们还可以方便地将这类数据分为“退化”(矢量型和欧几里得里奇) 和“非退化”(洛伦兹型里奇) 两类 (参见“正规顶点”一节)。

The explicit appearance of the Regge action in the large spin limit is particularly interesting. It is in fact known that the Regge action has a well-defined continuum limit given by the Einstein-Hilbert action, so that this result might be understood as a first step in proving that the model reproduces general relativity. However, the situation turns out to be more subtle.

大自旋极限中里奇作用量的明确出现尤为值得关注。事实上，众所周知里奇作用量具有由爱因斯坦-希尔伯特作用量给出的明确定义的连续极限，因此这一结果可以被看作证明该模型能重现广义相对论的第一步。然而实际情况要更为微妙。

One can extend the saddle point approximation to include the sums over the spins associated with an internal face, and this has been shown to imply that, in the limit of large boundary spins, the theory on a fixed cell complex is dominated by flat solutions only [91-97]. This shows that general relativity cannot be recovered from the theory by first taking the large spin limit and then taking the continuum limit. Indeed, from simple familiar examples, one might have expected this [98]. Rather, one must take both limits at the same time while ensuring a certain model-dependent inequality involving the curvature, the spin, and the Immirzi parameter holds in the process [99-102]. While this clarifies the way in which the classical limit must be taken, whether the correct classical limit can be obtained in this way with sufficient generality remains an open question.

我们可以扩展鞍点近似，将与内面关联的自旋求和包含进来，已有研究表明这意味着在大边界自旋极限下，固定胞腔复形上的理论仅由平坦解主导 [91-97]。这说明无法通过先取大自旋极限再取连续极限的方式从该理论中得到广义相对论，事实上，从我们熟悉的简单例子中就可以预见到这一点 [98]。正确的做法是必须同时取两个极限，并且在此过程中保证曲率、自旋和伊米尔齐参数之间满足一个依赖于模型的不等式 [99-102]。尽管这澄清了经典极限必须的取法，但能否在足够大的通用性下通过这种方式得到正确的经典极限仍然是一个开放问题。

Booster Decomposition and Numerical Evaluations

助推分解与数值计算

Numerical evaluations of (49) are difficult. It is a multidimensional unbounded integral over ratios of sums of hypergeometric functions which not only require high numerical accuracy but are furthermore rapidly oscillating. Furthermore, the results need to be tabulated for a large number of magnetic number combinations. It is significantly harder than numerics for the Barrett-Crane or SU(2)-based models.

(49) 式的数值计算十分困难，它是对超几何函数求和比值的多维无界积分，不仅要求很高的数值精度，被积函数还存在快速振荡。此外，结果需要针对大量磁量子数组合制表，计算难度远高于 Barrett-Crane 模型或基于 SU(2) 的模型。

One method to tackle this challenge is to exploit the decomposition of $SL(2,\mathbb{C})$ Clebsch-Gordan (CG) coefficients into SU(2) ones [73]. This allows one to rewrite the vertex amplitude (51) as a sum over SU(2) 15j symbols labeled by new virtual spins and intertwiners to be summed over:

解决这一难题的一种方法，是利用 $SL(2,\mathbb{C})$ 克莱布希-高登 (CG) 系数分解为 SU(2) CG 系数的性质 [73]。借此可将顶点振幅 (51) 改写为对 SU(2) 15j 符号的求和，SU(2) 15j 符号由需要求和的新虚自旋和交缠子标记:

$$A_v\left(j_f, i_e\right) = \sum_{l_f=j_f}^{\infty} \sum_{k_e} \{15j\}_{l_f,k_e} \prod_{e=2}^{5} B_4^{\gamma}\left(l_f, k_e; j_f, i_e\right). \tag{58}$$

The sums over the intertwiner labels k_e range over the usual CG inequalities, and

对交缠子标记 k_e 的求和满足常规 CG 不等式，且

$$B := \sum_{m_f=-j_f}^{j_f} \begin{pmatrix} l_f \\ m_f \end{pmatrix}^{(k_e)} \begin{pmatrix} j_f \\ m_f \end{pmatrix}^{(i_e)} \int_o^{\infty} dr \frac{\sinh^2 r}{4\pi} \prod_{f\in e} d^{(\gamma j_f, j_f)}_{l_f j_f m_f}(r) \tag{59}$$

are called ”booster” functions. The integrands $d^{(\rho,k)}$ are the reduced boost matrix elements [77].

被称为“助推”函数，被积函数 $d^{(\rho,k)}$ 是约化 boost 矩阵元 [77]。

The advantage of this method is that the unbounded integrals have been reduced to a single one per edge. Furthermore, the vertex amplitude is the much simpler SU(2) one, very fast to evaluate numerically. The price to pay is the introduction of new internal sums, which are unbounded and have to be truncated by hand in any numerical evaluation. The sum converges, but the speed of convergence depends on the configuration considered, and it is typically slower for Lorentzian boundary data, as opposed to Euclidean ones. Writing and optimizing numerical codes performing this numerical evaluation is a current focus of research [97, 103-105].

> 该方法的优势在于将无界积分化简为每条边对应一个无界积分，且顶点振幅为形式更简单的 SU(2) 顶点振幅，数值计算速度极快。代价是引入了新的内部求和，这类求和是无界的，所有数值计算中都需要手动截断。该求和是收敛的，但收敛速度取决于具体构型，和欧几里得边界数据相比，洛伦兹边界数据的收敛速度通常更慢。编写并优化完成该数值计算的数值代码是当前的研究重点 [97, 103-105]。

The booster decomposition is not the only approach to numerical evaluations: there exist by now a rich set of ideas and methods that have been developed to compute different aspects of the EPRL amplitude and more in general of spin foam models; see, e.g., [90, 100, 102, 106-109].

> 助推分解并非数值计算的唯一方法: 目前学界已经发展出了丰富的思路与方法，用于计算 EPRL 振幅乃至更一般的自旋泡沫模型的不同方面，参见例如文献 [90, 100, 102, 106-109]。

Extension to Non-simplicial Spin Foams: The KKL Model

非单形自旋泡沫的推广:KKL 模型

If one wants to give amplitudes to all possible spin network states, the first step is to define the model beyond 4-simplices. A nice way to systematically construct non-simplicial vertex amplitudes was studied in [53]. It was there applied to a straightforward extension of the EPRL model, based on applying the same set of constraints (48) to every link in the generalized vertex graph. The resulting amplitude which we refer to as the KKL model is the immediate generalization of (49) with the connectivity in the arguments of the representation matrices induced by the vertex graph.

> 如果要为所有可能的自旋网络态赋予振幅，第一步就是在 4-单形之外定义模型。文献 [53] 研究了一种系统构造非单形顶点振幅的简洁方法，该方法被应用于 EPRL 模型的直接推广，其核心是对广义顶点图的每条链接应用同一组约束 (48)。得到的振幅即我们所说的 KKL 模型，它是公式 (49) 的直接推广，其中表示矩阵参数的连通性由顶点图决定。

This generalization should, however, be taken with care, because it may be ill-defined depending on the graph. The convergence of the non-compact group integrals is not guaranteed by simply removing one redundant integration as in (49) and depends on the connectivity of the graph. A sufficient condition is 3-link-connectivity, namely, any bi-partition of the nodes cannot be disjointed by cutting only two links, but many non-3-link-connected graphs are well defined; see [110- 112]. This condition is satisfied by the 4-simplex graph.

但我们需要谨慎对待这一推广，因为它可能根据图的不同出现不良定义。非紧致群积分的收敛性无法像 (49) 那样仅通过移除一个冗余积分得到保证，而是依赖于图的连通性。三链接连通性是收敛的充分条件: 即对节点做任意二分化分都无法仅通过切断两条链接使两部分分离，但许多非三链接连通的图也是良定义的，参见 [110-112]。4-单形图满足这一条件。

The KKL model is appealing for its simplicity. Recall, however, that beyond the 4-simplex, the linear simplicity constraints are no longer sufficient to recover a discrete flat geometry around one vertex, already at the classical level. One may then expect the model to be dominated by a more general class of geometries than the Regge geometries appearing in the 4-simplex case. This is indeed what happens, and interestingly, the generalized geometries that appear have both a simple geometric description and admit a simple generalization of the Regge action [90,106,107,113].

KKL 模型的简洁性使其颇具吸引力。但需要注意，即使在经典层面，超出 4-单形后，线性简单约束也不足以还原单个顶点周围的离散平直几何。因此我们可以预期，该模型由比 4-单形情形下的里奇几何更广泛的一类几何主导。事实确实如此，有趣的是，出现的这些广义几何既有简洁的几何描述，还可以对里奇作用量做简单推广 [90,106,107,113]。

This is explained observing that the conditions required to obtain a second distinct saddle point are angle-matchings: they are equivalent to shape-matchings because once the areas of two triangles match, matching the angles identifies completely the triangles up to rotations. But matching the area and the angles of two n -sided polygons still leaves $n-3$ unmatched variables, corresponding to conformal transformations of the polygons. These conformal twisted geometries describe the critical behavior of the KKL model with distinct critical points. And the reason why the Regge action (57) is still well defined and indeed arises at the saddle points is that angle-matching is enough to have edge independence of the 4 d dihedral angles defined from the spherical cosine laws [113].

我们可以通过观察得到这一结论: 得到第二个不同鞍点所需的条件是角度匹配: 角度匹配等价于形状匹配，因为一旦两个三角形的面积匹配，角度匹配就能在旋转之外完全确定三角形。但匹配两个 n 边形的面积和角度后，仍然会留下 $n-3$ 个未匹配变量，对应多边形的共形变换。这些共形扭曲几何描述了 KKL 模型具有不同临界点时的临界行为。而里奇作用量 (57) 仍然良定义且恰好出现在鞍点的原因是，角度匹配足以保证由球面余弦定律定义的 4 d 二面角不依赖于边 [113]。

Applications of the KKL model include [112, 114, 115]. An attempt to modify the KKL model to include more constraints appeared in [116].

KKL 模型的应用参见 [112, 114, 115]。文献 [116] 尝试修改 KKL 模型以引入更多约束。

Proper Vertex

正则顶点

Consider a single 4-simplex v^* , with coherent boundary data $j_{ab} = j_{ba}, \mathbf{n}_{ab}$ for $a, b \in \{1, \dots, 5\}$. The expression for the vertex amplitude consists in an integral over the five $\mathrm{SL}(2, \mathbb{C})$ group elements $\{h_a\}$. Introduce an arbitrary flat connection ∂ inside the 4-simplex, such that the 4-simplex is the convex hull of its

0-simplices. When the critical point equations of the group integrals are satisfied, there exists a unique lie algebra valued 2-form B^{IJ} , constant with respect to ∂ , such that

考虑单个 4-单形 v^* ，它具有对应 $a, b \in \{1, \ldots, 5\}$ 的相干边界数据 $j_{ab} = j_{ba}, \mathbf{n}_{ab}$ 。顶点振幅的表达式是对五个 $\mathrm{SL}(2, \mathbb{C})$ 群元 $\{h_a\}$ 的积分。在 4-单形内部引入一个任意平坦联络 ∂ ，此时 4-单形是其 0-单形的凸包。当群积分的临界点方程满足时，存在唯一的李代数值 2-形式 B^{IJ} ，它关于 ∂ 是常数，满足

$$\int_{\Delta_{ab}} B^{IJ} = h_a \triangleright \left(j_{ab}(1,0,0,0) \wedge (0, \mathbf{n}_{ab})\right)^{IJ} \tag{60}$$

for all a, b , where Δ_{ab} denotes the triangle in v^* between tetrahedra a and b , oriented as part of the boundary of a , and $\triangleright$ denotes the adjoint action [117]. The resulting two-form B either is degenerate ($\mathrm{tr}(B \wedge \star B) = 0$) or is of the form $(I\pm)$ in (18) for some tetrad e in v^* , constant with respect to ∂ [117]. This is the precise sense in which the linear simplicity constraint, as imposed in the EPRL model, eliminates the sectors $(II\pm)$ in (18). For boundary data corresponding to a vector geometry or Euclidean Regge geometry, the critical point(s) selected corresponds to a two-form B that is degenerate, whereas for data corresponding to a Lorentzian Regge geometry, the two critical points selected correspond to the two sectors $(I\pm)$, giving rise precisely to the two exponentials in the large spin asymptotics equaling the cosine of the Regge action [75]. The existence of these multiple sectors is potentially problematic for a number of reasons:

对所有 a, b 成立，其中 Δ_{ab} 表示 v^* 中位于四面体 a 和 b 之间的三角形，定向为 a 边界的一部分，$\triangleright$ 表示伴随作用 [117]。由此得到的二形式 B 要么是退化的 ($\mathrm{tr}(B \wedge \star B) = 0$) ，要么形如 (18) 式中的 $(I\pm)$ ，对应 v^* 中某个关于 ∂ 为常数的标架 e [117]。这正是 EPRL 模型中施加的线性简单性约束排除 (18) 中 $(II\pm)$ 分支的准确含义。对于对应矢量几何或欧几里得里奇几何的边界数据，筛选出的临界点对应退化二形式 B ；而对于对应洛伦兹里奇几何的数据，筛选出的两个临界点对应两个分支 $(I\pm)$ ，恰好给出大自旋渐近中的两个指数，等于里奇作用量的余弦 [75]。这些多分支的存在从多方面来看都存在问题:

1. The degenerate sector, which classically must be excluded to recover general relativity, is present and not suppressed.

1. 经典层面上必须排除才能恢复广义相对论的退化分支确实存在，且未被压制。

2. Even if the degenerate sector were eliminated, the existence of two different nondegenerate sectors is potentially problematic, because there are no critical point equations forcing the sector to be the same across different 4-simplices. As a consequence, the actions appearing in the large spin limit of the amplitude, for sufficiently large triangulations, include sign flips as one goes from 4-simplex to 4-simplex, which changes the corresponding equations of motion so that they no longer approximate those of general relativity. This contrasts with the classical theory, where elimination of the degenerate sector is enough to ensure that the solution is either entirely in the sector $(I+)$ or entirely in the sector $(I-)$. (That being said, there are choices of triangulation and boundary data that force the sector to be uniform, relaxing this issue in such cases [102].)

2. 即使退化分支被消除，两个不同非退化分支的存在仍存在问题，因为没有临界点方程约束不同 4-单形必须属于同一分支。因此, 对于足够精细的三角剖分, 当我们从一个 4-单形过渡到另一个 4-单形时，振幅大自旋极限中出现的作用量会发生符号翻转，这会改变相应的运动方程，使其不再近似广义相对论的运动方程。这与经典理论不同，经典理论中排除退化分支就足以保证整个解要么完全属于分支 ($I+$)，要么完全属于分支 ($I-$)。(话虽如此，存在三角剖分和边界数据的选择可以强制分支统一，缓解这类情况下的问题 [102]。)

3. Results from the analysis of 3d gravity [118] suggest that such non-gravitational equations of motion, by allowing for unsuppressed "spikes," are furthermore responsible for bubble divergences in spin foam models with zero cosmological constant, such as EPRL. Such divergences have so far impeded calculations involving sufficiently refined triangulations.

3. 三维引力的分析结果 [118] 表明，这种非引力运动方程允许未被压制的“尖峰”，还会进一步导致零宇宙常数自旋泡沫模型 (如 EPRL 模型) 中出现泡发散。这类发散迄今为止阻碍了涉及足够精细三角剖分的计算。

In addition, it has been suggested that these multiple sectors are an obstruction to an exact match with the canonical theory [119] and that restriction to a single sector may be analogous to the positive frequency condition in loop quantum cosmology necessary to extract correct physics [120]. A similar restriction has also been advocated by Oriti [121] as a way of implementing causality in the sense introduced by Teitelboim [122].

此外，已有研究指出这些多分支是与正则理论精确匹配的障碍 [119]，并且限制到单个分支可能类似于圈量子宇宙学中提取正确物理所必需的正频率条件 [120]。Oriti 也支持类似的限制，将其作为实现 Teitelboim[122] 引入的因果性的一种方式 [121]。

For these reasons, it is interesting to see if it is possible to remove the degenerate and ($II-$) sectors from the vertex amplitude, so that only the ($II+$) sector remains. This is the idea behind the "proper vertex," in which a classical inequality is first derived selecting the ($II+$) sector, which then is quantized as a projector and inserted into the EPRL vertex amplitude [117, 123]. Results so far for the proper vertex reproduce the successful tests of the EPRL vertex amplitude [124, 125], as well as provide indications that some of the above issues might be resolved [126, 127]. One drawback of the proper vertex is that it is less simple than the ERPL vertex, suggesting that it may be worthwhile to seek a simpler alternative that achieves the same goals.

因此，探究是否可以从顶点振幅中剔除简并 sector 和 ($II-$) sector，仅保留 ($II+$) sector 是一项有意义的工作。这就是“真顶点”背后的核心思路: 先推导一个经典不等式筛选出 ($II+$) sector，再将其量子化为投影算符，插入 EPRL 顶点振幅中 [117, 123]。目前真顶点的研究结果复现了 EPRL 顶点振幅的成功检验结果 [124, 125]，同时也有迹象表明上述部分问题可能得到解决 [126, 127]。真顶点的一个缺点是它比 EPRL 顶点更复杂，这说明寻找一个能实现相同目标的更简单的替代方案是值得的。

Extension to Non-space-Like Building Blocks

推广到类非空间积木

Within the framework of Regge calculus, it is possible to approximate arbitrarily well a pseudo-Riemannian geometry using space-like tetrahedra, and this is the reason why most models are built this way. However, it is also true that allowing 4-simplices with non-space-like faces and tetrahedra allows a richer sampling of the continuum spacetime, as well as more symmetric choices of shapes for simple geometries. With such motivations in mind, it is possible to consider a version of the BC and EPRL models with not only space-like but also time-like faces; see, e.g., [49, 128-130]. A particularly interesting result in this generalization is that a 4-simplex amplitude with mixed faces does not admit vector geometries as critical points; hence, its semiclassical limit is dominated by Regge geometries alone [131]. Using null-faces on the other hand remains more elusive [59, 81, 132, 133].

在里奇微积分框架下，我们可以用类空间四面体任意良好地逼近伪黎曼几何，这也是多数自旋泡沫模型都这样构建的原因。但事实证明，允许包含带类非空间面和类非空间四面体的 4 单形，不仅能更丰富地对连续时空采样，还能为简单几何选取更具对称性的形状。带着这些研究动机，我们可以考虑一种同时包含类空间面和类时间面的 BC 模型与 EPRL 模型，参见例如文献 [49, 128-130]。这种推广中一个特别有意思的结论是，带混合面的 4 单形振幅不将向量几何作为临界点，因此其半经典极限仅由里奇几何主导 [131]。另一方面，使用零界面的研究目前仍无明确结论 [59, 81, 132, 133]。

Inclusion of Cosmological Constant

包含宇宙学常数

The basic idea of the EPRL model, and other spin foam models before it, is to start from $SL(2,\mathbb{C})$ quantum BF theory on a simplicial complex and then impose the simplicity constraints at the tetrahedra in order to obtain gravity, imitating Plebanski at the quantum level. Besides boundary labels being restricted by simplicity, the vertex amplitude of EPRL is simply the vertex amplitude of BF theory, and the flatness equation of motion of BF theory (28) is the reason why each 4-simplex is flat in the large spin limit. The most recent and well-developed extension of EPRL to include a cosmological constant [134-137] builds on this basic approach, except that, instead of starting from BF theory, it starts from a modification of BF theory, which one may call ”Holst- ΛBF ,” that includes the cosmological constant in a way that also involves the Immirzi parameter. Specifically, the action is that in Equation (15) with the Lagrange multiplier ϕ removed. When the form of B in the $(I\pm)$ sectors of the solution to simplicity is substituted into this action, it reduces to the Holst action of gravity with Newton constant $\pm G$ and Immirzi parameter γ , with an extra term giving cosmological constant $\pm\Lambda$, where the signs depend on the sector. Variation of the above action with respect to B yields the equation of motion

EPRL 模型以及此前其他旋泡沫模型的基本思路是: 从单纯形复形上的 $SL(2,\mathbb{C})$ 量子 BF 理论出发, 在四面体处施加简单性约束以得到引力, 在量子层面模仿 Plebanski 理论。除边界标记受简单性约束限制外，EPRL 的顶点振幅就是 BF 理论的顶点振幅，而 BF 理论的平坦运动方程 (28) 就是每个 4-单形在大自旋极限下平坦的原因。目前最成熟、最新的包含宇宙学常数的 EPRL 扩展方案 [134-137] 建立在这一基本思路上，区别仅在于该方案并非从 BF 理论出发，而是从 BF 理论的一种修正形式出发，可称之为“霍尔斯特- ΛBF ”，这种形式引入宇宙学常数的同时也包含了伊米尔齐参数。具体来说，该作用量就是方程 (15) 中去掉拉格朗日乘子 ϕ 后的形式。将简单性解 $(I\pm)$ 区中 B 的形式代入该作用量后，它就约化为带有牛顿常数 $\pm G$ 和伊米尔齐参数 γ 的引力霍尔斯特作用量，外加一个给出宇宙学常数 $\pm\Lambda$ 的额外项，符号由所选区决定。对上述作用量关于 B 变分得到运动方程

$$F = \frac{\Lambda}{3}B \tag{61}$$

so that the solutions are no longer flat as in (28), but of ”constant curvature” in the sense above. In sectors $(I\pm)$,(61) imposes constant Ricci scalar curvature $\pm 4\Lambda$.

因此解不再像 (28) 中那样是平坦的，而是具有上述意义下的“常曲率”。在 $(I\pm)$ 区中，(61) 给出常里奇标曲率 $\pm 4\Lambda$ 。

The lack of flatness is also reflected in the continuum path integral. Since the action is no longer linear in B , but quadratic, one no longer obtains a Dirac delta function imposing flatness as in (31), but rather a Gaussian integral, leaving as integrand the exponential of an action with Lagrangian quadratic in F equaling a total derivative, so that the action reduces to an action on the 3d boundary, specifically the Chern-Simons action. The level of the Chern-Simons theory is complex, given by $\kappa = \frac{12\pi}{8\pi\Lambda\ell_P^2}\left(\frac{1}{\gamma}+i\right)$, with $\ell_P^2 := G\hbar$ the Planck area. Invariance of the above expression under large gauge transformations forces $\mathrm{Re}\,\kappa \in \mathbb{Z}$, which forces the cosmological horizon area, $12\pi/|\Lambda|$, to be an integer multiple of $8\pi\gamma G\hbar$ - that is, that it corresponds to a loop quantum gravity area eigenvalue with integer spin $\mathrm{Re}\,\kappa$!

非平坦性也体现在连续路径积分中。由于作用量不再是 B 的线性函数，而是二次函数，我们不再得到像 (31) 中那样施加平坦性约束的狄拉克 δ 函数，反而得到一个高斯积分，其被积函数是一个作用量的指数，该作用量的拉格朗日量关于 F 是二次的，且等于一个全导数，因此作用量约化为 3 维边界上的作用量，具体来说就是陈-西蒙斯作用量。陈-西蒙斯理论的能级是复数，由 $\kappa = \frac{12\pi}{8\pi\Lambda\ell_P^2}\left(\frac{1}{\gamma}+i\right)$ 给出，其中 $\ell_P^2 := G\hbar$ 是普朗克面积。上述表达式在大规范变换下的不变性要求 $\mathrm{Re}\,\kappa \in \mathbb{Z}$ ，这就要求宇宙视界面积 $12\pi/|\Lambda|$ 是 $8\pi\gamma G\hbar$ 的整数倍——即它对应于整数自旋 $\mathrm{Re}\,\kappa$ 的圈量子引力面积本征值!

The Chern-Simons theory on the boundary of each 4-simplex is topological and so yields a finite number of degrees of freedom at each vertex. By excluding the links of the boundary graph from the theory, the resulting quantum degrees of freedom again match those of loop quantum gravity on the boundary graph, except with spins j bounded above by $\mathrm{Re}\,\kappa/2 \in \mathbb{Z}/2$.

每个 4-单形边界上的陈-西蒙斯理论是拓扑的，因此在每个顶点给出有限个自由度。将边界图的链排除出该理论后，得到的量子自由度再次与边界图上圈量子引力的自由度一致，区别仅在于自旋 j 存在上界 $\mathrm{Re}\,\kappa/2 \in \mathbb{Z}/2$ 。

As one can see, this extension has many beautiful surprises and leads to a completely new way to look at the theory. Moreover, the fact that it includes a cosmological constant is consistent with what we observe in our universe so far. It is interesting, and not necessarily problematic, that both signs of the cosmological constant are included in the theory.

不难看出，这一扩展带来了许多漂亮的新发现，开辟了看待该理论的全新路径。此外，该理论包含宇宙学常数这一点也符合我们目前对宇宙的观测结果。值得一提的是，该理论同时包含正负两种符号的宇宙学常数，这一结论很有意思，也未必存在问题。

The fact that one obtains an upper bound on the spins is reminiscent of how quantum deformations of classical Lie groups can make the number of irreps finite. However, since the level κ is complex, no relation of the above model to a quantum group has yet been established. Studies of 3 d gravity have long suggested a relation between the inclusion of a cosmological constant and the use of quantum groups, starting from the Turaev-Viro model [28]. See, for example, [138] and references therein. Attempts to use quantum groups to incorporate a cosmological constant also into 4d spin foam models include [139-141].

自旋存在上界这一事实，让人联想到经典李群的量子形变可以使不可约表示的数量有限。但由于能级 κ 是复数，目前尚未建立上述模型与量子群的关联。早在图拉耶夫-维罗模型 [28] 提出时，对 3 d 引力的研究就长期表明，引入宇宙学常数和使用量子群之间存在关联，相关讨论参见例如文献 [138] 及其中引文。也已有不少工作尝试将量子群用于在 4 维旋泡沫模型中引入宇宙学常数，包括文献 [139-141]。

Conclusions

结论

The spin foam formalism provides a specific framework to define the path integral for quantum gravity so that its boundary states are the spin networks of the Hilbert space of loop quantum gravity. The goal of the program is to find a spin foam amplitude that implements correctly the diffeomorphism constraints and reduces to general relativity in the appropriate limit. This would provide an ultraviolet complete quantum theory of gravity, whose geometrical and physical interpretation can be studied using results from loop quantum gravity on geometric operators.

自旋泡沫形式为定义量子引力路径积分提供了一个特定框架, 使得它的边界态就是圈量子引力希尔伯特空间的自旋网络。该研究方案的目标是找到能够正确实现微分同胚约束、并在适当极限下退化为广义相对论的自旋泡沫振幅。这将给出一个紫外完备的量子引力理论，利用圈量子引力在几何算符方面的研究结果，我们就可以研究该理论的几何与物理解释。

Currently, the most studied model is the EPRL model, of which we described here in some details the Lorentzian version. This model has various promising properties: it provides transition amplitudes to all spin network states, it includes a non-trivial dependence on γ , it passes a test on 4-simplex correlations failed by the BC model, and it has a clear relation to the Regge action in the large spin limit. This last result has been often invoked as a first step toward a proof that the model reproduces general relativity, since it is known that the Regge action reduces to the Einstein-Hilbert action in the continuum limit. However, the situation is more complicated. As noted at the end of section ”Large Spin Asymptotics”, the large spin limit of the theory on a fixed cell complex is unrelated to general relativity, so that this limit must be combined appropriately with a refinement limit. Moreover, as discussed in sections ”Large Spin Asymptotics” and ”Proper Vertex”, the Regge action appears in the large spin limit of the amplitude only for non-degenerate configurations, and there exist degenerate configurations whose amplitude has similar fall-off and so are not suppressed. Furthermore, even for non-degenerate configurations, the Regge action appears twice, once for each sign, independently for each 4-simplex, leading to actions for larger triangulations with alternating signs and no relation to gravity. If these latter features turn out to indeed prevent the correct classical limit of the model, the proper vertex discussed in section ”Proper Vertex” offers one possible modification to remove them. Beyond the key question of the semiclassical limit, numerous applications of the EPRL model have appeared in the literature, including the graviton propagator [142-144], quantum tunneling for the black-hole-to-white-hole transition [145, 146], and spin foam cosmology [112,114,115,120,147,148].

目前，研究最多的模型是 EPRL 模型，本文已对其洛伦兹版本进行了详细介绍。该模型拥有诸多颇具前景的性质: 它给出了所有自旋网络态的跃迁振幅，包含对 γ 的非平凡依赖，通过了 BC 模型未能通过的 4 单形关联检验，并且在大自旋极限下与 Regge 作用量有明确关联。这最后一项结果常被视为证明该模型可以重现广义相对论的第一步，因为已知 Regge 作用量在连续极限下退化为爱因斯坦-希尔伯特作用量。然而实际情况更为复杂。正如“大自旋渐近分析”一节末尾所述，固定胞腔复形上该理论的大自旋极限与广义相对论无关，因此必须将该极限与精化极限适当结合。此外，正如“正顶点”一节的讨论，仅在非简并构型下，振幅的大自旋极限才会出现 Regge 作用量，而存在一些简并构型，其振幅也具有相似的衰减性，因此不会被压低。更进一步，即使是对于非简并构型，Regge 作用量也会以两种符号各出现一次，且每个 4 单形的符号相互独立，这导致大三角剖分对应的作用量符号交替，与引力无关。如果这些特征确实会阻碍模型得到正确的经典极限，那么“正顶点”一节讨论的正顶点就是一个可以去除这些问题的可能修改方案。除了半经典极限这一核心问题之外，文献中已经出现了 EPRL 模型的诸多应用，包括引力子传播子 [142-144]、黑洞到白洞跃迁的量子隧穿 [145, 146] 以及自旋泡沫宇宙学 [112,114,115,120,147,148]。

In this chapter, we focused on the EPRL model, in 3+1 dimensions and the BF-plus-constraints approach, but in closing, we would like to stress and convey that the spin foam formalism provides a valuable framework for studying covariant transition amplitudes of quantum geometry more generally. The conceptual and technical framework we reviewed here can be applied and extended to models with different signature, different dimensions or including matter sectors. It provides an arena for studying renormalization and continuum limit in a background-independent manner.

本章我们聚焦于 3+1 维 EPRL 模型以及 BF 加约束框架，但在结尾我们需要强调: 更一般地说，自旋泡沫形式为研究量子几何的协变跃迁振幅提供了极有价值的框架。我们在这里回顾的概念与技术框架可以应用并拓展到不同符号差、不同维度或是包含物质部分的模型中，它为以背景独立的方式研究重整化和连续极限提供了平台。

Open Questions and Research Directions

开放问题与研究方向

Among the open question and research directions that are being investigated, we can mention the following ones.

在目前正在研究的开放问题与研究方向中，可列举如下。

- In three dimensions, gravity is a topological field theory, and the only degrees of freedom appear on the boundary. The corresponding spin foam construction gives the Ponzano-Regge and Turaev-Viro models, depending on the value of cosmological constant. The inclusion of boundary degrees of freedom in these models has been recently investigated and led to a promising convergence of results with other approaches to quantum gravity [149, 150].

- 三维引力是拓扑场论，仅在边界存在自由度。根据宇宙学常数取值的不同，相应的旋量泡沫构造得到庞扎诺-雷格模型与图雷耶夫-维罗模型。近来已有研究探讨如何在这些模型中纳入边界自由度，所得结果与其他量子引力研究方法趋于一致，前景可观 [149, 150]。

- Explicit couplings to various types of matter have been written down [151-155], but there is still work to do to understand the resulting matter dynamics, for instance, whether one can see explicitly the regularization of UV divergences as has been argued for from the canonical perspective [24].

- 旋量泡沫模型已经写出了与多种物质的显式耦合 [151-155]，但要理解由此产生的物质动力学仍有很多工作要做，例如能否像正则量子引力视角所论证的那样，显式地看到紫外发散的正规化 [24]。

- Spin foam models are ultraviolet finite, but the EPRL model and its predecessors do have infrared divergences, in particular, so-called bubble divergences that prevent calculation for sufficiently large triangulations. From studies of 3d gravity, it has been suggested that such divergences are due to diffeomorphism invariance [156]. However, on closer inspection, the configurations leading to the divergence are not diffeomorphism related, but correspond to ”spikes” with non-zero deficit angles that are not isometric to any region in $\mathbb{R}^3$, so that they represent violations of the classical equations of motion, a violation which can be traced to the independent summing over orientations in each tetrahedron, the ”cosine problem” [118]. The proper vertex discussed above provides one possible solution to this. When the cosmological constant is included, all sums are finite merely because the range of spins is finite, but it is not clear whether the underlying physical problem remains.

- 旋量泡沫模型在紫外是有限的，但 EPRL 模型及其前身确实存在红外发散，尤其是所谓的泡发散，导致无法对足够大的三角剖分进行计算。三维引力的研究表明，这类发散源于微分同胚不变性 [156]。但进一步考察发现，导致发散的构型与微分同胚无关，而是对应带有非零亏缺角的“尖峰”，这类构型不与 $\mathbb{R}^3$ 中的任意区域等距，因此它们代表对经典运动方程的违背，这种违背可追溯到对每个四面体的取向独立求和，即“余弦问题”[118]。前文讨论的合适顶点为此提供了一种可能的解决方案。引入宇宙学常数后，由于自旋取值范围本身有限，所有求和都是有限的，但背后的物理问题是否仍然存在尚不清楚。

- The derivation of the spin foam sum from the projector onto solutions of the Hamiltonian constraint (see section ”Transition Amplitudes for Spin Network States”) leads to a sum over all 2-complexes. An open question is whether such a sum over 2-complexes is equivalent to considering a fixed 2-complex and then taking its refinement limit. Work is in progress to understand how the amplitudes scale when refining the cell complex [107, 112, 157] .

- 从哈密顿约束解的投影算子推导旋量泡沫求和 (参见“自旋网络态跃迁振幅”一节) 会得到对所有 2 复形的求和。一个开放问题是，这种对所有 2 复形的求和是否等价于先固定 2 复形再取其加细极限。目前正在开展研究，以理解加细胞复形 [107, 112, 157] 时振幅的标度行为。

- As reviewed in section ”Transition Amplitudes for Spin Network States”, a motivation for the spin foam framework comes from expanding the projector onto solutions of the Hamiltonian constraint operator. However, this derivation is incomplete, and the spin foam ansatz is not exactly taken from it. As a consequence, the relation of the EPRL and other spin foam models to canonical quantizations of the Hamiltonian constraint is an open issue. Work in this direction include, for example, [119, 158, 159].

- 正如“自旋网络态跃迁振幅”一节所回顾的，旋量泡沫框架的动机来自对哈密顿约束算符解投影算子的展开。但这一推导并不完整，旋量泡沫猜想也并非严格由此得出。因此，EPRL 模型及其他旋量泡沫模型与哈密顿约束正则量子化的关系仍是一个开放问题，该方向的相关研究包括例如 [119, 158, 159]。

- The KKL extension on an arbitrary vertex lacks some of the classical simplicity constraints, and it is an open question how to add them and whether they successfully reduce the dominant contributions from conformal twisted geometries to Regge geometries [90, 106, 107, 113, 116].

- 在任意顶点上的 KKL 扩展缺少部分经典简单性约束，如何添加这些约束，以及它们能否成功将共形扭几何的主导贡献约化为里奇几何 [90, 106, 107, 113, 116]，仍是一个开放问题。

- Even on a single 4-simplex, the imposition of the primary simplicity constraints alone may fail to take into account the modification to the quantum measure that originates from the standard treatment of quantizing systems with second-class constraints [160].

- 即使仅在单个四 simplex 上，只施加原生简单性约束也可能无法考虑到，对带二类约束系统量子化标准处理中产生的量子测度修正 [160]。

- Conversely, it has been argued in [101] that including second-class constraints in a quantum system with spacetime discreteness makes the emergence of semiclassical physics more subtle than simply taking the $\hbar \to 0$ limit. This observation has led to the construction of effective spin foam models based on the imposition of simplicity constraints in terms of discrete geometries, as opposed to a restriction of the quantum labels. The effective models offer a handle on the interplay between refining and the emergence of curvature and also a useful link to area-Regge calculus with a new perspective on the recovery of general relativity [161].

- 反过来，文献 [101] 指出，在具有时空离散性的量子系统中纳入二类约束会使得半经典物理的涌现比单纯取 $\hbar \to 0$ 极限更为微妙。这一观察推动了有效旋量泡沫模型的构造，这类模型基于按离散几何施加简单性约束，而非限制量子标号。有效模型为理解加细与曲率涌现之间的相互作用提供了切入点，也与面积-里奇微积分建立了有用联系，并为恢复广义相对论提供了新视角 [161]。

- The BF quantization used as starting point to impose the simplicity constraints depends only on data on a 2-complex. This has been called into question in [162], arguing that additional data on 3-cells is needed on manifolds of nontrivial topology.

- 用作施加简单性约束起点的 BF 量子化仅依赖于 2 复形上的数据。文献 [162] 对此提出了质疑，认为在具有非平凡拓扑的流形上，还需要三维胞上的额外数据。

Take-Home Message

核心要点

The foundations of the spin foam formalism have multiple roots. From implementing the constraints via a path integral formulation, generalizing the mathematical framework of topological quantum field theories, and providing a sampling of quantum geometries, numerous ideas have converged toward an ansatz like (13). Explicit realizations of this idea have been constructed in the literature and spurred many mathematical developments and applications. The current open questions and research directions offer a stimulating window on all facets of the quantum gravity problem, and this provides one of the key interests to keep working in the spin foam approach.

旋子泡沫形式体系的基础有多重来源。从通过路径积分表述实现约束，推广拓扑量子场论的数学框架，到给出量子几何的抽样形式，诸多思想汇聚形成了式 (13) 这类 ansatz。这一思想的显式实现已在文献中被构建，并推动了大量数学发展与应用。当前的开放问题与研究方向为量子引力问题的各个层面提供了富有启发性的视角，这也是持续开展旋子泡沫方法研究的核心吸引力之一。

Cross-References

交叉引用

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黎曼量子几何的涌现

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- Philosophical Foundations of Loop Quantum Gravity

- 圈量子引力的哲学基础

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自旋泡沫、精修极限与重整化

Spinfoams and High-Performance Computing

自旋泡沫与高性能计算

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